An Exploration of $n$-Trigonometric Functions: Generalizations of Sine and Cosine
Alex Beckwith ’14 with Dr. Marie Snipes, Department of Mathematics, Kenyon College, Gambier, Ohio

Introduction

For a complex-valued function $f$ analytic in the unit disk, if $f'(0) = f''(0) = f^{(3)}(0) = \ldots = 0$, then $f$ is an even function. This is apparent from the Taylor expansion

$$f(z) = f(0) + \frac{f'(0)}{1!}z + \frac{f''(0)}{2!}z^2 + \frac{f^{(3)}(0)}{3!}z^3 + \ldots;$$

since each of the summands has even power, we get an even function. In particular, the complex cosine function is even:

$$f(z) = \cos(z) = \sum_{j=0}^{\infty} (-1)^j \frac{z^{2j}}{(2j)!} = \cos(-z).$$

If we differentiate termwise we find the familiar pattern of differentiation:

$$f'(z) = -\sin(z), \quad f''(z) = -\cos(z), \quad f^{(3)}(z) = \sin(z),$$

Note that cosine satisfies the differential equation $f''(z) = -f$.

Abstract

In this project we begin by considering the differential equation $f''(z) = -f$ for a fixed $n$. We use the so-called power series method to identify $n$ solutions to this differential equation. These functions are called the $n$-trigonometric functions (thus, sine and cosine are 2-trigonometric functions). In this project we study geometric and analytic properties of the $n$-trigonometric functions.

Finding $n$-Trigon Functions

Notice that in the Taylor series for cosine, only even powers of $z$ appear. We can construct solutions to $f''(z) = -f$ from a Taylor series where the only powers that appear are multiples of three. We construct the series as follows:

$$3z_0(z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(3j)!} z^{3j}$$

$$3z_1(z) = \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) \sum_{j=0}^{\infty} \frac{(-1)^{j+1}}{(3j+2)!} z^{3j+2}$$

$$3z_2(z) = \left(\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) \sum_{j=0}^{\infty} \frac{(-1)^{j+1}}{(3j+1)!} z^{3j+1}$$

Note from the power series for $3z_0$ that if we take derivatives, then we cycle through multiples of $3z_0$ and $3z_2$. Additionally, $3z_0$ has a property of symmetry, with

$$3z_0(z) = 3z_0\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)z = 3z_0\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)z.$$

The functions $3z_1$ and $3z_2$ are also symmetric in this fashion. In the general setting, we can define analytic functions possessing similar properties for any natural number $n \geq 1$ using similar methods.

Definition: For a natural number $n$, we define the principal $n$-trigonometric function by

$$n_{z_0}(z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(nj)!} z^{nj},$$

and we define the nonprincipal $k$th $n$-trigonometric function by

$$n_{z_k}(z) = e^{k \pi i /n} \sum_{j=0}^{\infty} \frac{(-1)^j}{(nj + k)!} z^{nj+k},$$

with $k = 1, 2, ..., n - 1$.

Closed Form Expressions

To find closed forms for the $n$-trigonometric functions, we began by considering the solution space of the differential equation $f''(z) = -f$. Using the power series method, we found that the solutions of $f''(z) = -f$ are linear combinations of the $n$-trigonometric functions:

$$f(z) = \sum_{k=0}^{n-1} c_k \cdot e^{k \pi i /n} \cdot n_{z_k}$$

where $c_0, ..., c_{n-1}$ are constants. We were also able to obtain a generalization of Euler’s formula:

$$e^{x e^{i \pi/n}} = \sum_{k=0}^{n-1} n_{z_k}(x).$$

For the principal $n$-trigonometric function, we were able to obtain the closed form expression

$$n_{z_0}(z) = \sum_{k=0}^{n-1} e^{e^{i k \pi /n}} W_{n,k}$$

from which closed forms for the nonprincipal functions may be obtained. We omit those here. Note that the closed form for $n_{z_0}$ is an average of exponential functions; for example, for $n_{z_0}$ we have

$$n_{z_0}(z) = e^{-z} + e^{\left(1 + \frac{2\sqrt{3}}{3}\right) z} + e\left(\frac{2\sqrt{3}}{3}\right) z.$$