An Exploration of n-Trigonometric Functions: Generalizations of Sine and Cosine

Alex Beckwith '14 with Dr. Marie Snipes, Department of Mathematics, Kenyon College, Gambier, Ohio

Introduction

For a complex-valued function f analytic in the unit disk, if $f'(0) = f^{(3)}(0) = f^{(5)}(0) = \cdots = 0$, then f is an even function. This is apparent from the Taylor expansion

$$f(z) = f(0) + \frac{f^{(2)}(0)}{2!}z^2 + \frac{f^{(4)}(0)}{4!}z^4 + \cdots;$$

since each of the summands has even power, we get an even function. In particular, the complex cosine function is even: $f(z) = \cos(z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} z^{2j} = \cos(-z)$. If we differentiate termwise we find the familiar pattern of differentiation:

$$f'(z) = -\sin(z) \qquad f^{(3)}(z) = \sin(z)$$
$$f''(z) = -\cos(z) \qquad f^{(4)}(z) = \cos(z)$$

Note that cosine satisfies the differential equation $f''(z) = -f$.

Abstract

In this project we begin by considering the differential equation $f^{(n)} = -f$ for a fixed n. We use the so-called power series method to identify n solutions to this differential equation. These functions are called the n-trigonometric functions (thus, sine and cosine are 2-trigonometric functions). In this project we study geometric and analytic properties of the n-trigonometric functions.

Finding n-Trig Functions

Notice that in the Taylor series for cosine, only even powers of z appear. We can construct solutions to $f^{(3)} = -f$ from a Taylor series where the only powers that appear are multiples of three. We construct the series as follows:

$$_3t_0(z) := \sum_{j=0}^{\infty} \frac{(-1)^j}{(3j)!} z^{3j}$$
$$_3t_1(z) := \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \sum_{j=0}^{\infty} \frac{(-1)^{j+1}}{(3j+2)!} z^{3j+2}$$
$$_3t_2(z) := \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \sum_{j=0}^{\infty} \frac{(-1)^{j+1}}{(3j+1)!} z^{3j+1}$$

Note from the power series for $_3t_0$ that if we take derivatives, then we cycle through multiples of $_3t_1$ and $_3t_2$. Additionally, $_3t_0$ has a property of symmetry, with

$$_3t_0(z) = {}_3t_0\left(\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)z\right) = {}_3t_0\left(\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)z\right).$$

The functions $_3t_1$ and $_3t_2$ are also symmetric in this fashion. In the general setting, we can define analytic functions possessing similar properties for any natural number $n \geq 1$ using similar methods:

Definition: For a natural number n, we define the **principal n-trigonometric function** by

$$_nt_0(z) := \sum_{j=0}^{\infty} \frac{(-1)^j}{(nj)!} z^{nj},$$

and we define the nonprincipal **kth n–trigonometric function** by

$$_nt_k(z) := e^{k\pi i/n} \sum_{j=0}^{\infty} \frac{(-1)^j}{(nj+k)!} z^{nj+k},$$

with $k = 1, 2, \ldots, n-1$.

Complex Maps and Riemann Surfaces

Because of the two-dimensionality of complex variables, it is impossible to visualize complex-valued functions in the familiar graphical way. Instead, we rely on representations that describe the images of regions of the complex plane under a given complex-valued function. For the n-trigonometric functions, our work is simplified by the fact that they satisfy a symmetry property; we need only consider the image of certain portions of the complex plane to obtain the range of the function. The example maps that follow are all plots for

$$_4t_0(z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(4j)!} z^{4j}.$$

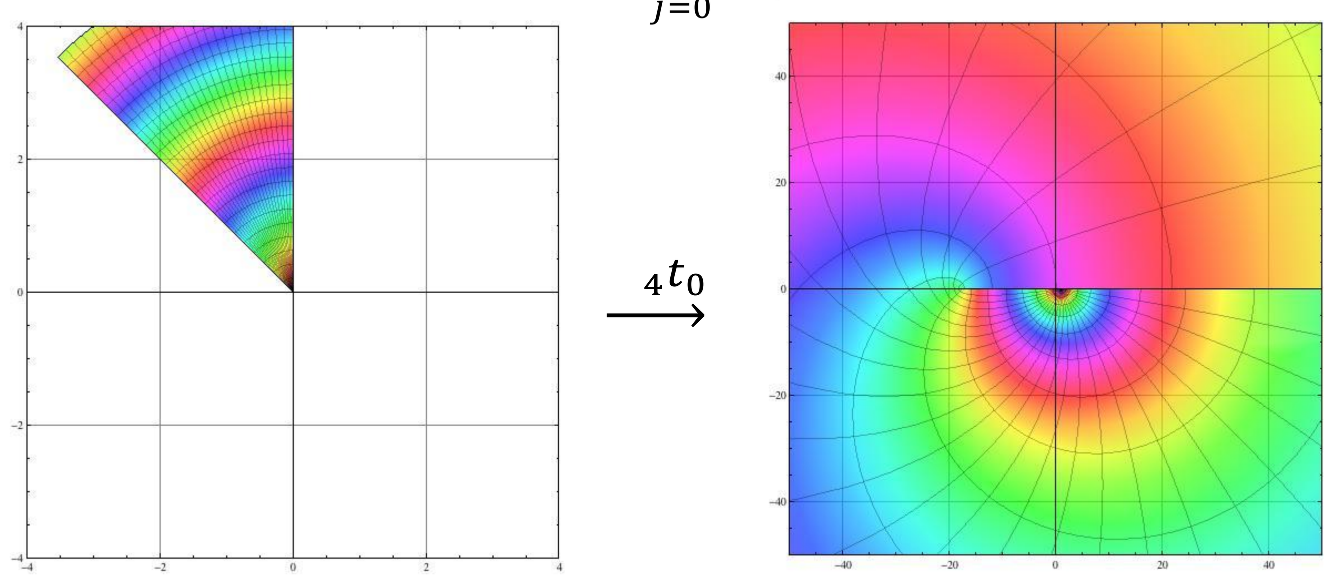
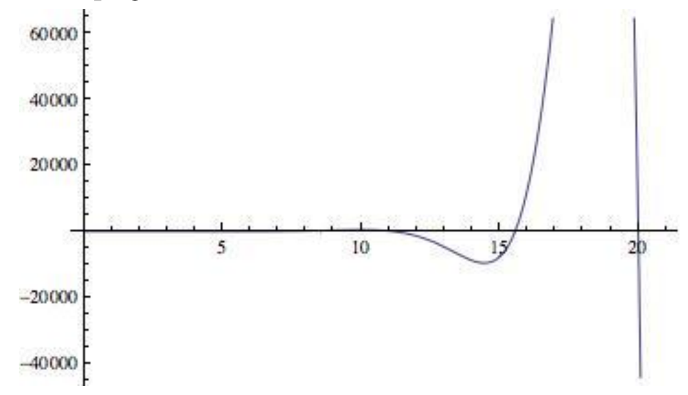


Figure 1. An region and its image under $_4t_0$. The colors and angles of the first graph are preserved under $_4t_0$.

Furthermore, because $_4t_0$ has a convergent power series expansion, it is conformal (angles are preserved). Hence, we note that there is a "wrapping" around the real line, evidenced by the image of the real-line under $_4t_0$:



Because of the symmetry property, however, $_nt_0$ is multivalued, and so we require a Riemann surface to unambiguously define its inverse. We determined experimentally the fundamental region for $_4t_0$ to be approximately as follows:

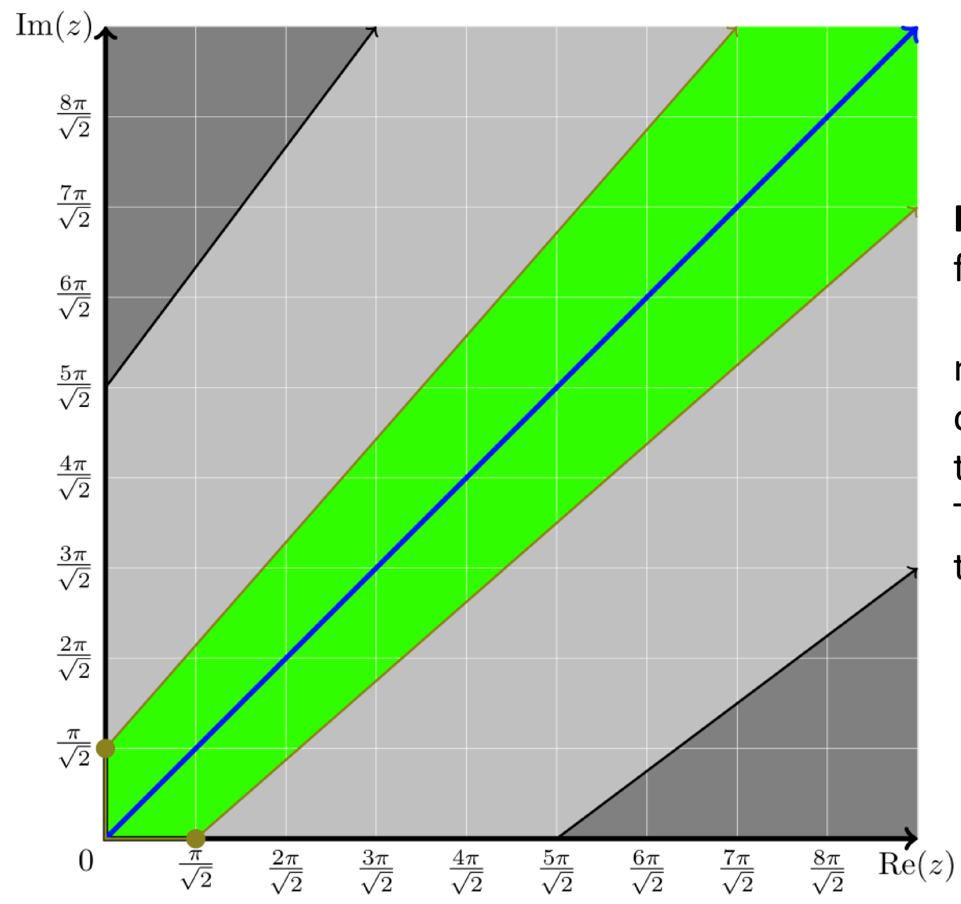


Figure 2. Approximate fundamental regions for $_4t_0$. The area in green maps to the entire complex plane, as will the areas shaded in gray. The blue line maps to the real line.

Closed Form Expressions

To find closed forms for the n-trigonometric functions, we began by considering the solution space of the differential equation $f^{(n)} = -f$. Using the power series method, we found that the solutions of $f^{(n)} = -f$ are linear combinations of the n-trigonometric functions:

$$f(z) = \sum_{k=0}^{n-1} k! \cdot c_k \cdot e^{-\frac{k\pi i}{n}} \cdot {}_nt_k,$$

where $c_0, \ldots, c_{n-1}$ are constants. We were also able to obtain a generalization of Euler's formula:

$$e^{z \cdot e^{i\pi/n}} = \sum_{k=0}^{n-1} {}_nt_k(z).$$

For the principal n-trigonometric function, we were able to obtain the closed form expression

$$_nt_0(z) = \frac{1}{n} \sum_{k=0}^{n-1} e^{z \cdot e^{i(\pi + 2k\pi)/n}},$$

from which closed forms for the nonprincipal functions may be obtained. We omit those here. Note that the closed form for $_nt_0$ is an average of exponential functions; for example, for $_3t_0$ we have

$$_3t_0(z) = \frac{e^{-z} + e^{\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)z} + e^{\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)z}}{3}.$$

Because both the familiar trigonometric functions and the hyperbolic functions share a close relationship with the exponential function, we can express the n-trigonometric functions in terms of familiar functions. For example, for n divisible by 4 we have

$$_nt_0(z) = \frac{4}{n} \cdot \sum_{j=0}^{\frac{n-4}{4}} \cos\left(z \cdot \sin\left(\frac{2j+1}{n}\pi\right)\right) \cdot \cosh\left(z \cdot \cos\left(\frac{2j+1}{n}\pi\right)\right).$$

Generalized Hypergeometric Functions and Ongoing Work

Definition: A **generalized hypergeometric function** is a function that may be expressed as a power series where the ratios of successive coefficients is a rational function. Using rising factorial notation, we may express this as

$$_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) := \sum_{j=0}^{\infty} \frac{(a_1)_j \ldots (a_p)_j}{(b_1)_j \ldots (b_q)_j} \cdot \frac{z^j}{j!}.$$

Theorem: The n-trigonometric functions may be expressed as generalized hypergeometric functions as follows:

$$_nt_k(z) = {}_0F_{n-1}\left(;\frac{1}{n} + 1, \ldots, \frac{k}{n} + 1, \frac{k+1}{n}, \ldots, \frac{n-1}{n}; -\frac{z^n}{n^n}\right).$$

Generalized hypergeometric functions have numerous connections with number theory. In particular, they have an interesting relationship with continued fractions and generalizations of continued fractions called G-continued fractions. For example, using the properties of $_0F_1$ we can obtain the continued fraction expansion of Gauss for tangent:

$$\tan(z) = \cfrac{z}{1 - \cfrac{z^2}{3 - \cfrac{z^2}{5 - \cfrac{z^2}{7 - \ddots}}}}.$$

Our goal in expressing n-trigonometric functions as generalized hypergeometric functions is to exploit some of the well-studied properties of the latter.

References

[1] Saff, E. B., and Snider, A. D. *Fundamentals of Complex Analysis with Applications to Engineering and Science*. – Prentice Hall, Upper Saddle River, New Jersey, 2003.

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