Minimal Length Scale and Its Effect on Hawking Radiation

Patrick Meyers ‘12, Demian Cho
Department of Physics, Kenyon College, Gambier, OH

Abstract

The effect on the tunneling rate of a quantum particle moving across the Rindler horizon was calculated in the presence of a minimal length scale. This is a toy model for Hawking Radiation, where the Rindler space (the space of properly accelerating observers) horizon represents a black hole horizon, and tunneling particles represent the radiation. This form of calculating radiation in Rindler space recovers the same result for the temperature of thermal particles emitted from a black hole that was calculated by Hawking in 1974. Using a generalized form of the Heisenberg Uncertainty Principle to implement a minimal length scale, the change in this tunneling rate in Rindler space was calculated as a change to the imaginary action of a particle traveling through Rindler space. The final calculation yielded a change in the imaginary action relating to the square of the Planck length, the energy of the tunneling particle, and the acceleration of that particle. While the correction is small (the Planck length is on the order of 10^{-35}m), it posits a correction nonetheless. Moreover, due to red shifting of waves near the horizon of a black hole it could produce more significant results for observers far from the horizon itself. Future work will include generalizing this to the full Schwarzschild metric, instead of a toy model, to provide a more formal setting for the tunneling.

Introduction to Black Holes

A black hole is formally defined as a region of space from which no information can be communicated outward. An observer inside the black hole can not communicate with an observer outside of the black hole. The observer in the black hole is said to be causally disconnected from any observer outside the black hole. This would then imply that nothing should come out of a black hole, including light.

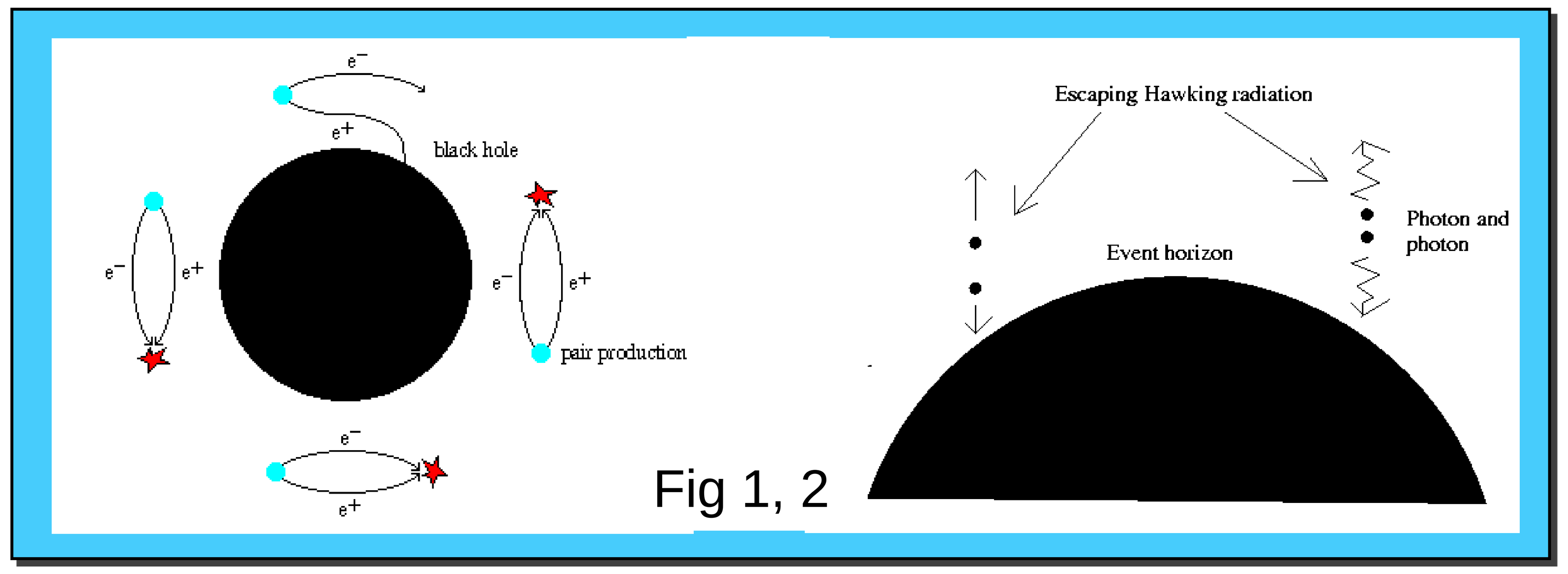


Fig 1, 2

Hawking Radiation and Tunneling

In 1974 Stephen Hawking used a semiclassical approximation to show that black holes should radiate thermal particles. While the formal method of calculating this radiation was not done using quantum mechanical tunneling, even Hawking himself used it as a convenient intuitive picture. The intuition comes from imagining pair production of a particle and anti-particle (a known consequence of the uncertainty principle) in the region close to a black hole horizon. If we assume that one particle falls into the black hole while the other escapes (Fig. 1,2), then we would have particles appearing to emanate from the black hole. Moreover, in formal particle physics an anti-particle is a particle moving backwards in time, and so the anti-particle (“anti” relative to the particle that escapes!) moving into the black hole can be represented as a particle moving *out* of the black hole. However, based on the definition of a black hole as not allowing anything (including light) to escape, this is classically not allowed. Therefore, quantum mechanical tunneling must be introduced to explain this phenomenon.

Fig. 3, 4

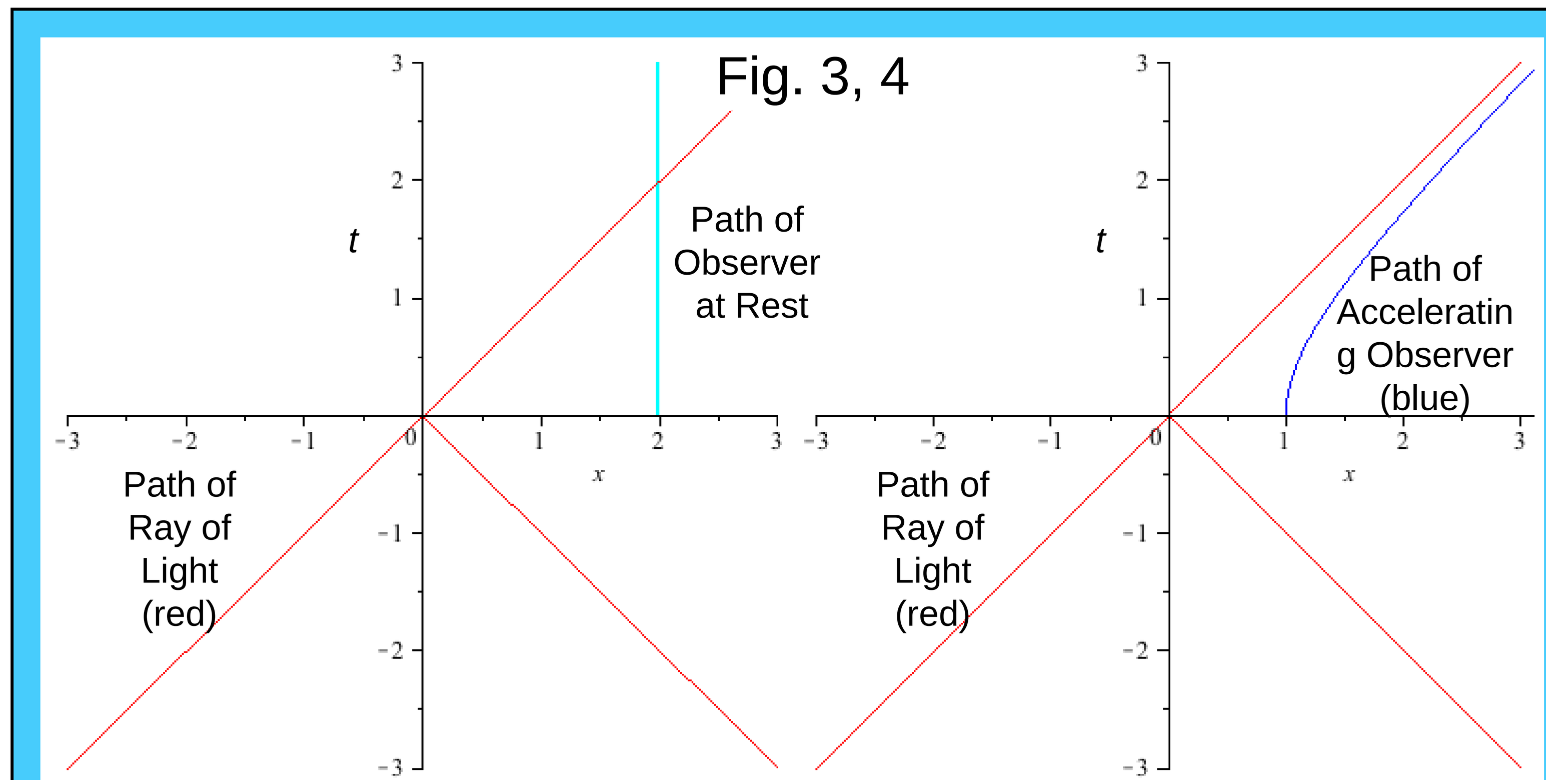
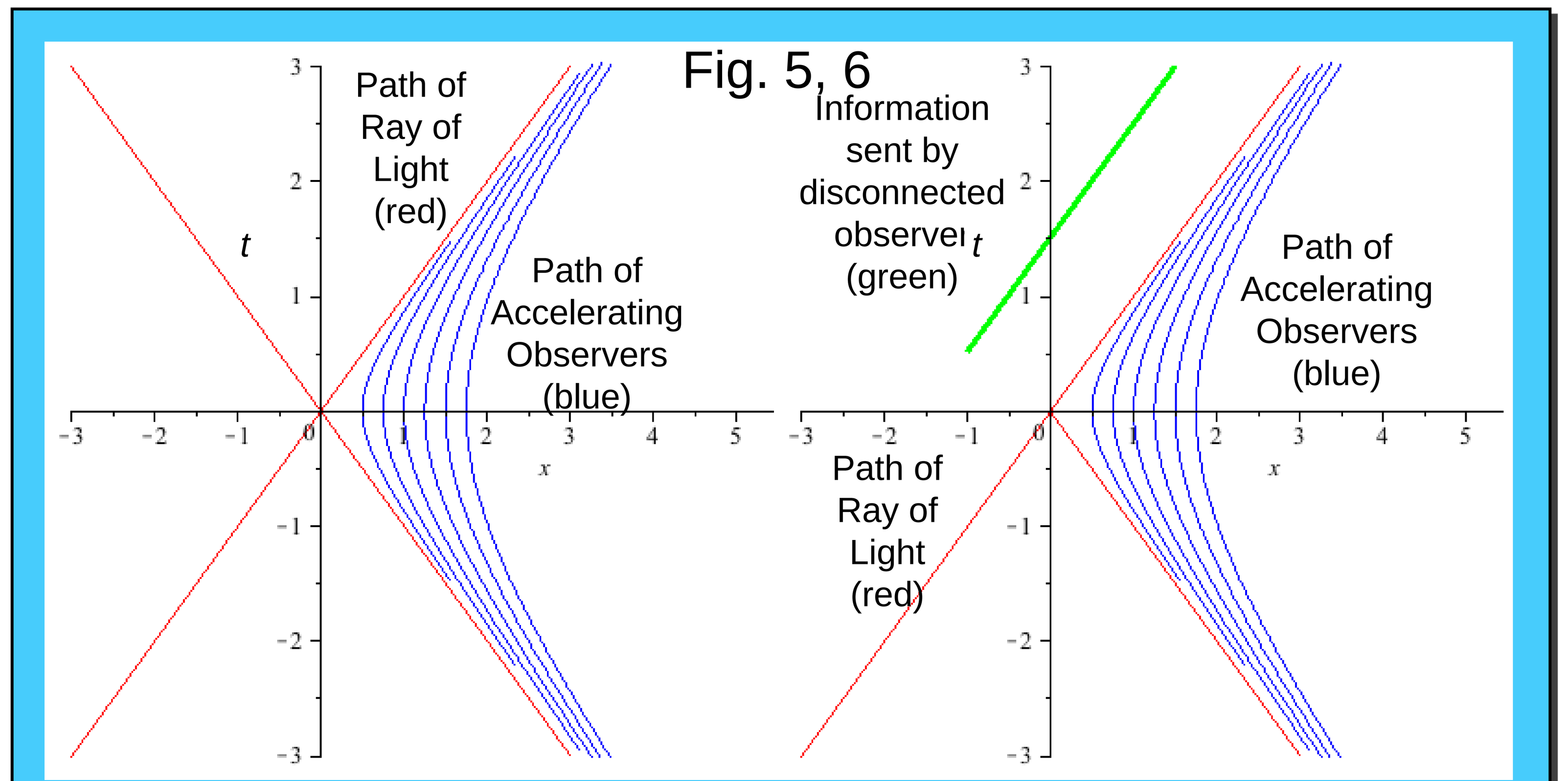


Fig. 5, 6



Rindler Spacetime

In 1+1 (one spatial plus one time) dimensional spacetime, an observer at rest in space moves only through time (straight vertical line, Fig. 3). Consider an observer with constant acceleration starting at rest at time t=0: initially the observer will look like it's moving only through time and as time goes on its worldline should look more and more like that of the speed of light, approaching it asymptotically (Fig. 4). This can then be extended back in time to consider the observer decelerating in the negative x direction, coming to rest at t=0 and some distance x_0 from the origin before moving off as before (Fig 5). In this regime, the distance x_0 is the inverse of the acceleration of the observer (in units where the speed of light is one).

Clearly, observers in the left part of the spacetime can't communicate with observers to the right (Fig 6). They are causally disconnected from observers in the right hand part of the space. This makes Rindler spacetime a good model for a black hole, and one that is much easier to deal with mathematically than formal black hole physics.

These accelerating observers need a method for measuring distances. The normal spacetime interval does not hold for accelerating observers. Parameterizing the motion of the accelerating observers and using this to create a new metric we find the new metric (where x_R and t_R are lines of constant position and time for each observer).

$$ds^2 = -(1+2ax_R)dt_R^2 + (1+2ax_R)^{-1}dx_R^2$$

Hawking Radiation in Rindler Spacetime

To consider Hawking radiation as tunneling across the Rindler horizon, a method was devised to calculate a quantum mechanical tunneling coefficient in the absence of a formal potential energy barrier. In classical mechanics it is the action, *S*, that determines the trajectory of a particle in space. Classically, the action is not well-defined for imaginary values; that is, imaginary action relates to a classically disallowed region for the particle. In quantum mechanics, particles in classically disallowed areas are ones that have tunneled. By assuming that the classical action of the particle is the one that satisfies the Klein-Gordon equation, we extend that action to take imaginary values and relate the imaginary part of the action to the probability of tunneling across the barrier. We make the ansatz that the wavefunction, psi, satisfying the Klein-Gordon equation and the tunneling coefficient, T, have the form below [1][2].

$$\psi = e^{\frac{i}{\hbar}S} \qquad T = e^{-2\,\mathrm{Im}S}$$

The form of the Klein-Gordon equation (shown below, with units of the speed of light as one and with hats representing canonical operators) is inherently dependent upon the metric of the space in which the calculation is done, and here the Rindler metric becomes important. This calculation was done by Andrea de Gill et. al [1], and the original Hawking temperature of the emitted radiation was recovered. This summer's research was spent reproducing this calculation in the setting of a minimal length scale.

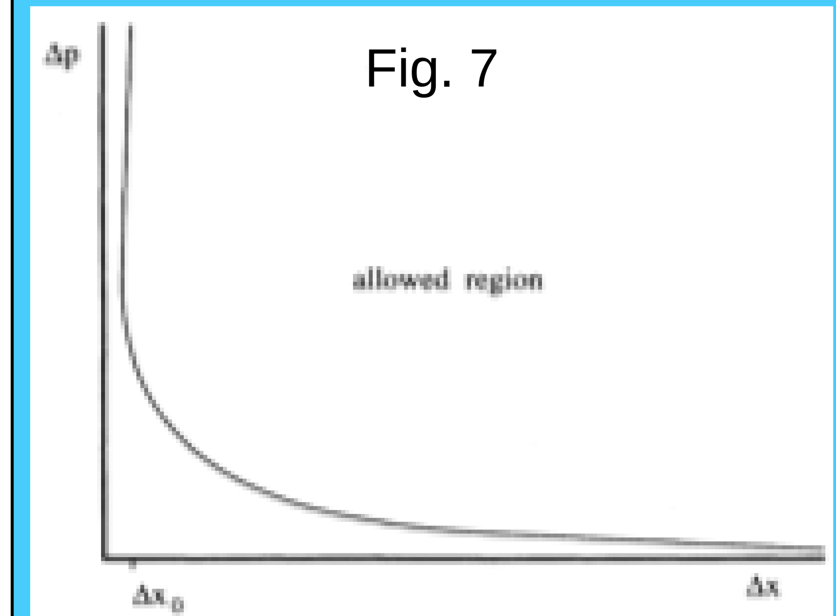
$$\hat{p}^2\psi + \frac{m^2}{\hbar^2}\psi = \frac{\partial^2\psi}{\partial t^2} \quad \text{(Klein Gordon Equation)}$$

Minimal Length Scale and Generalized Uncertainty Principle

The motivation from a minimal length scale comes from the idea that using light to probe small distances requires the wavelength of light to be on the order of that distance. This then requires a very high amount of energy in a very small space. Past a certain length, a black hole is created; effectively cutting off anything on that length scale from communicating with anything on a larger one. The method used to implement a minimal length scale in this project was to consider a minimum uncertainty in the *x* direction. If it is impossible to resolve past a certain length, then uncertainties in length or position can't have infinitesimally small values either. Motivated by this, a modification was made to Heisenberg's Uncertainty Principle to not allow infinitesimally small values in the uncertainty in *x*. This modification is shown below (Fig. 7), along with a graph (Fig 5) showing how the uncertainty in *x* is bounded below. [3][4]

$$\Delta x \Delta p \geq \frac{\hbar}{2} \qquad \Delta x \Delta p \geq \frac{\hbar}{2}\left[1+\beta(\Delta p)^2\right]$$

Fig. 7



Implementing the Generalized Uncertainty Principle

To implement the new uncertainty principle into quantum mechanics, a change in the canonical momentum was made to achieve the necessary commutation relations that give the uncertainty principle displayed above. The quantization of momentum was only in the spatial dimensions and was

$$p_i = p_{0i}(1-\beta p_0^2) \quad \text{where} \quad p_0^2 = \sum_{j=1}^{3} p_{0j}p_{0j} \quad \text{and} \quad \beta = \frac{l_p^2}{2\hbar^2}$$

The Klein-Gordon equation was used to evaluate the action of a particle in Rindler space. However, the Klein-Gordon equation is a relativistic equation that inherently depends upon the momentum of the particle. Therefore, a change in the canonical momentum operator changes the Klein-Gordon and results in a new action that satisfies it.

Results

When the action was solved for using the modified Klein Gordon equation and semiclassical ansatz as before, the solution came in two parts. The first was the same imaginary part of the action as de Gill et. al that produced the Hawking Temperature. The second part came as a result of the implementation of the new uncertainty principle and was a correction to the action calculated by de Gill et al. The additional imaginary part of the action was found to be

$$\mathrm{Im}S = l_p^2 \pi E a$$

Where lp is the Planck length, E is the energy of our particle, and a is the acceleration of the observer in Rindler spacetime. This constitutes a correction to the tunneling coefficient and hence the Hawking temperature itself.

References

[1] A. de Gill et. al. *American Journal of Physics.* 2010, 78, 685-691.
[2] K. Srinivasan; T. Padmanabhan. *Physical Review D*. 1999, 60, 024007-1 - 024007-20.
[3] A. Kempf; G. Mangano; R. B. Mann. *Physical Review D*. 2009, 52, 1108-1118.
[4] S. Das; E. C. Vanegas. arXiv:0901.1768. 2009, 1-7.

Acknowledgments

I would like to thank Professor Cho for directing and helping me in this research, as well as the Kenyon Summer Science Scholars program for sponsoring it.