A Geometric Representation of the Abundancy Index

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**Background**

Some of the oldest open problems in mathematics involve perfect numbers. These numbers are integers whose sum of proper divisors equals the number itself. More formally, a positive integer, \( n \), is said to be perfect if the sum of its divisors, \( \sigma(n) \), is equal to \( 2n \). In Elements, Euclid proved that if \( 2^n - 1 \) is a prime number (a Mersenne prime), then \( 2^{n-1}(2^n - 1) \) is a perfect number. Close to two thousand years later, Euler showed conversely that the even perfect numbers are exactly those of this form. The smallest perfect number is 6, and there are 47 known perfect numbers, corresponding to the known Mersenne primes. In fact, it is still unknown whether there are infinitely many Mersenne primes, and equivalently, even perfect numbers. Moreover, even the existence of an odd perfect number is unknown.

**Abundancy Outlaws**

It turns out, there are rational numbers that are not abundancy indices for any number. These fractions are called abundancy outlaws. Understanding when abundancy outlaws occur is important for determining the conditions necessary for the existence of an odd perfect number.

Weiner’s Outlaws: If \( \frac{a}{b} \) is in lowest terms and \( k < h < \sigma(k) \), then \( \frac{k}{h} \) is an abundancy outlaw. For instance, \( \frac{1}{2} \) is an abundancy outlaw since \( 5 < \sigma(4) = 7 \).

If we follow the same process that we used to color the lines of fractions with denominators 4 and 5, we get the following picture. Below, we also update our picture by coloring the bricks corresponding to rational numbers that are abundancy outlaws red.

**Prime Denominators**

Notice that we cannot yet identify any rational numbers that have prime denominators as abundancy outlaws. If we recreate our geometric representation by considering only rational numbers with prime denominators, we get the following picture. Notice that the \( n^\text{th} \) row for the bottom corresponds to the \( n^\text{th} \) prime number (rather than \( n \) itself).

**Generalization of Patterns**

In general, if \( p \) and \( \frac{1}{p} - 1 \) are both prime, then \( I(k) = \frac{p}{p-1} \) are of this form. This suggests that any rational number in this range that does not satisfy the criteria above is an abundancy outlaw.

**Discussion**

Classifying sequences of rational numbers as abundancy outlaws is a very difficult problem. This stems from the fact that we must prove that a particular form of rational number cannot be an abundancy index for any number. Even still, the abundancy index function can give us insight into the forms of the indices that do occur. As part of our research this summer, we created a geometric representation of a set of rational numbers to help us find these patterns. As a result, we have a better idea of when abundancy indices with prime denominators occur and conjectures for when they cannot occur.

**References and Acknowledgments**

**References:**


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