Numerical Techniques of Integration for Volumes of Revolution

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Abstract

In Calculus one learns how to approximate the definite integral with left, right, midpoint, and trapezoid Riemann Sums. The error of these sums can be described in terms of the first and second derivatives of the integrand, f. The definite integral describes the area under the graph of f. Our project explores numerical techniques for approximating the volume obtained when f is rotated about the x axis. We define left, right, midpoint, trapezoid, and Simpson approximations for this setting. Then we examine the error for these methods and create bounds.

Method

Defining Volume Sums: Right and Left Volume sums are well defined and commonly used. However, Trapezoid, Midpoint, and Simpson’s Rule are not as well defined because their equivalent average of sums notation does not hold equivalent in Volume settings. Thus these weighted sums for area are completely disregarded for Volumes of Revolution. Instead we will use Newton’s method for interpolating polynomials to define the Trapezoid and Simpson’s Rule, and Taylor Polynomials to define the Tangential Midpoint Rule.

Finding Error Bounds for Volume Sums: We start by understanding the computation for the maximum error bound for left Riemann sums. Let us make a simplified function of $f(x)$ called $K(x)$ that goes through the point, $(a, f(a))$ and whose derivative at any point in the interval (a, b) does not exceed the maximum first derivative of $f(x)$. $K(x)$ is called a “worst case” function. Observe that $K(x)$ has the same left approximation as $f(x)$. Also note how the error for $K(x)$ is larger than the error for $f(x)$. Therefore, if we compute the area of the error for $K(x)$, which is conveniently a right triangle, we will have a maximum error bound for area. We can rotate the error about the x axis to get the error bound for Volumes of Revolution. To apply this to n partitions, we must take into consideration n and the distance the function is from the x axis at the left endpoint. We use a similar geometric method for the other sums.

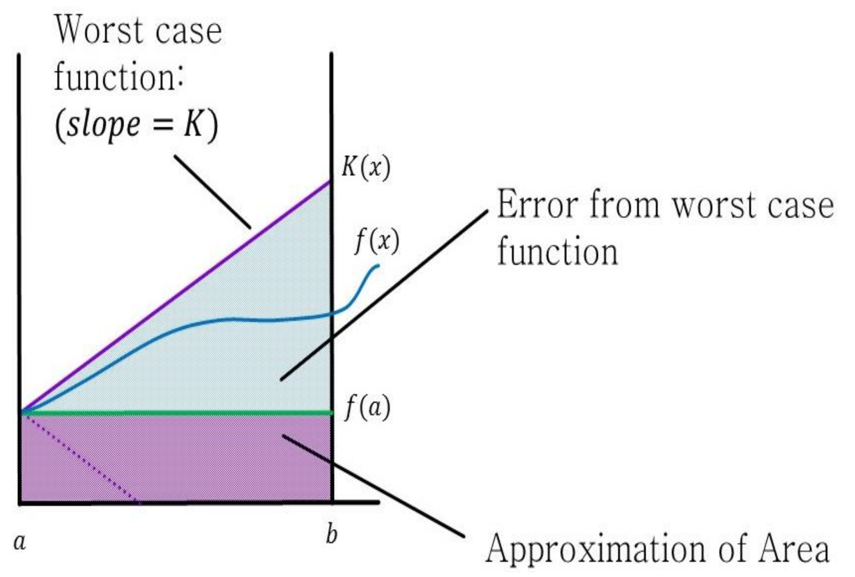

Figure I: A geometric method of how to compute error bounds for a Left Riemann Sum Approximation

Results

Defining Volume Sums:
Over a given interval $[a, b]$, let the x axis be partitioned in n equal subdivisions where i is the i^{th} subdivision. Let $\Delta x = \frac{b-a}{n}$. Let $x_{i-1} = (i-1)\Delta x + a$, the left endpoint. Let $x_i = i\Delta x + a$, the right endpoint. Let $m_i = \frac{x_{i-1}+x_i}{2}$, the midpoint. Let these volume sums be rotated about the x axis.

Tangential Midpoint $$:= \sum_{i=1}^{n} \pi \int_{x_{i-1}}^{x_i} (f(m_i) + f'(m_i)(x - m_i))^2 dx$$

Trapezoid Volume Sum $$:= \sum_{i=1}^{n} \left(\frac{\pi(f(x_i)^3 - f(x_{i-1})^3)}{3(f x_i) - f(x_{i-1}))} \right) \Delta x$$

Simpson’s Rule $$:= \sum_{i=1}^{n} \frac{2\pi}{105} (58 f(x_i)^2 + 3f(x_i)f(x_{i-1}) + 21 f(x_i) f(x_{i+1}) + 23 f(x_{i-1})^2 - 14 f(x_{i-1}) f(x_{i+1}) + 14 f(x_{i+1})^2) \Delta x$$

Defining Error Bounds for Volume Sums:
Let K be a constant such that $|f'(x)| \leq K$ for all x in $[a, b]$. Let M be a constant such that $|f''(x)| \leq M$ for all x in $[a, b]$.

Left Error $$\leq \pi K f(a)(b-a)^2 + \frac{\pi K^2 (b-a)^3}{3}$$

Right Error $$\leq \pi K f(a)(b-a)^2 + \frac{\pi K^2 (b-a)^3}{3}$$

Midpoint Error $$\leq \frac{\pi K M (b-a)^4}{48 n^3} + \frac{\pi K^2 (b-a)^3}{12 n^2} + \frac{\pi K (b-a)(Area\ under\ f(x))}{2n}$$

Tangential Midpoint Error $$\leq \frac{\pi M^2 (b-a)^5}{144 n^4} + \frac{\pi M K}{6 n^3} + \frac{\pi M (b-a)^2 (Area\ under\ f(x))}{3 n^2}$$

Trapezoid Error $$\leq \frac{2\pi M^2 (b-a)^5}{15 n^4} + \frac{\pi M K}{6 n^3} + \frac{\pi M (b-a)^2 (Area\ under\ f(x))}{3 n^2}$$

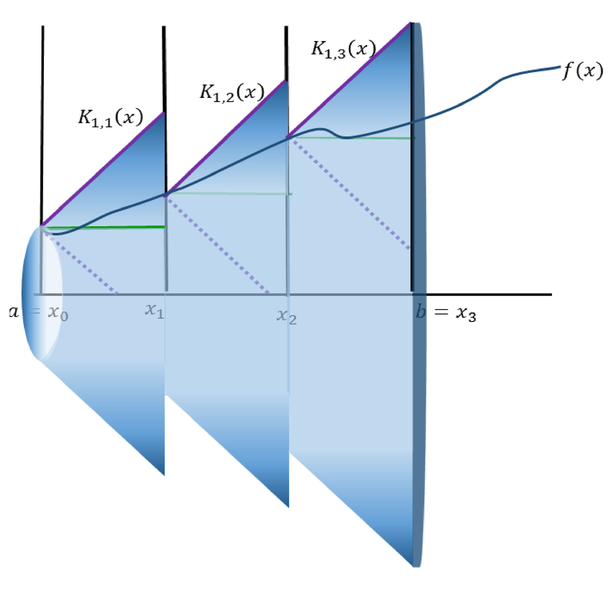

Figure II: Volume Error for a Right Volume Sum with $n = 3$.

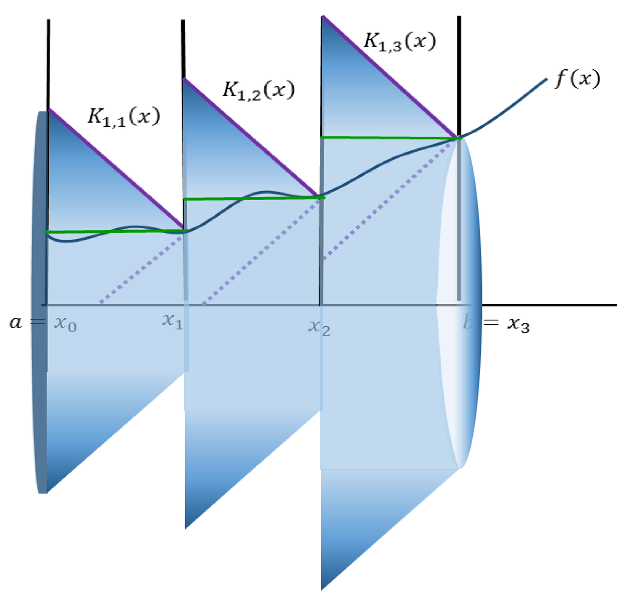

Figure III: Volume Error for a Left Volume Sum with $n = 3$.

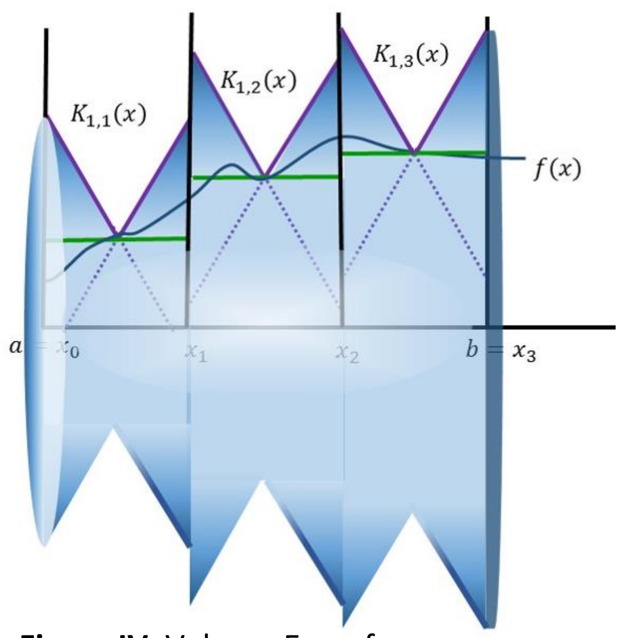

Figure IV: Volume Error for a Midpoint Volume Sum with $n = 3$.

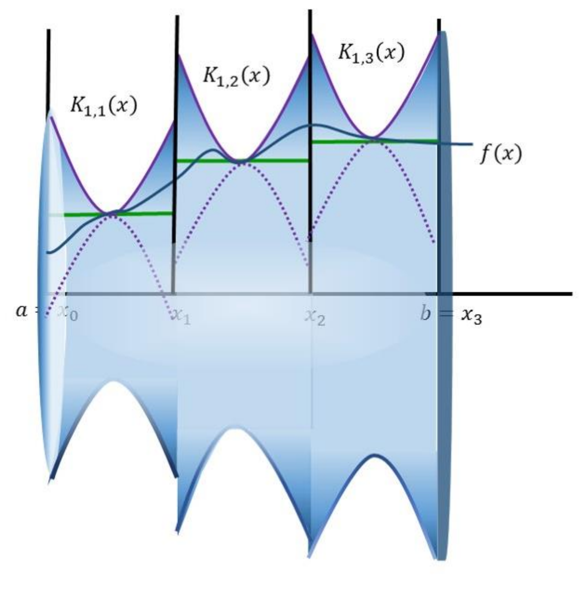

Figure V: Volume Error for a Tangential Midpoint Volume Sum with $n = 3$.

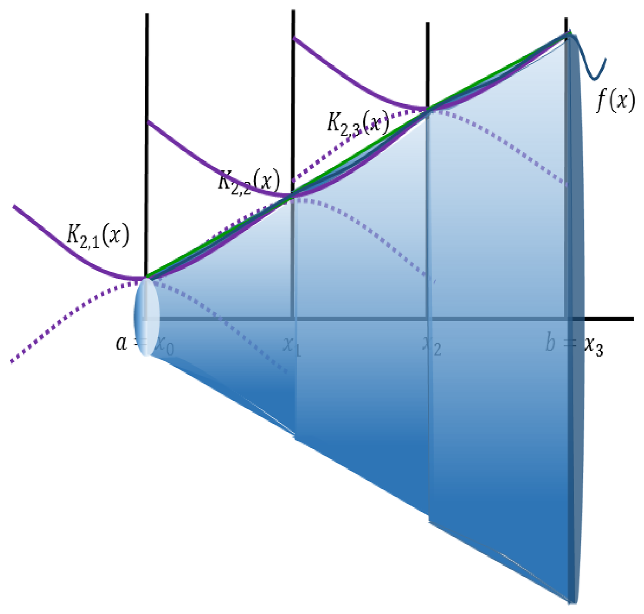

Figure VI: Volume Error for a Trapezoid Volume Sum with $n = 3$.

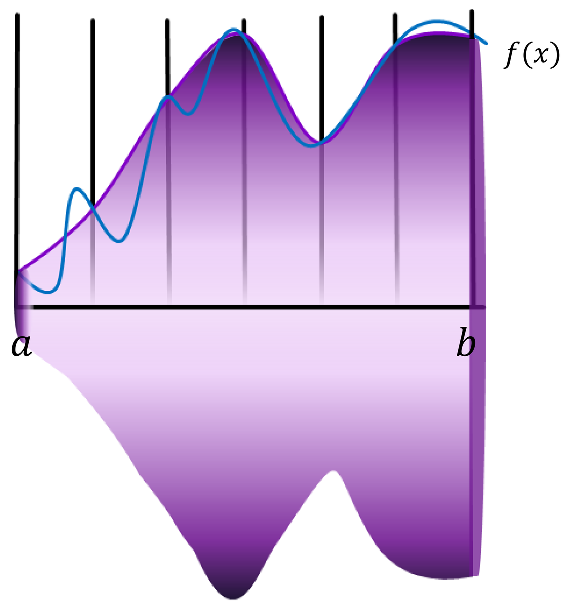

Figure VII: Defining Volume Sums for Simpson’s Rule when $n = 6$.

Discussion

It is important to note that there are two factors that influence error for volume approximations. One being the steepness (Right, Left), concavity (Midpoint, Trapezoid), or how far away the function is from being a degree three polynomial (Simpson’s). The other being how far the function is from the axis of rotation. The figure below illustrates this for a left volume approximation. The second factor does not affect error in area approximations. Thus our bounds had to take this factor into consideration, whereas area error bounds do not. Hence we have the $Area\ under\ f(x)$ term in our error bounds. Because this is a maximum error bound, it sufficient to provide an over-approximation for this term.

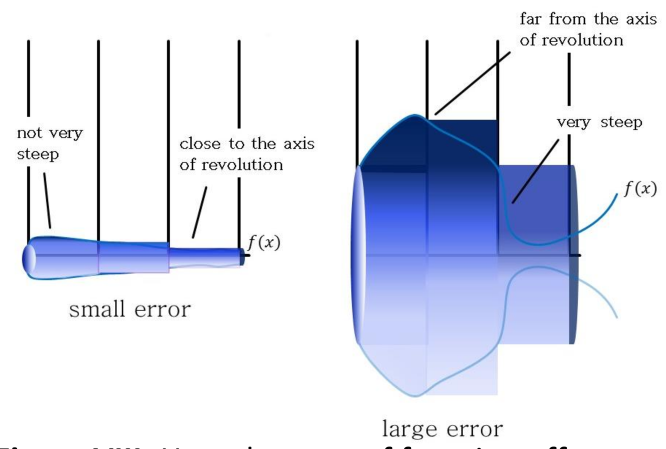

Figure VIII: How the type of function affects error (blue shaded region) for a Left Volume Sum

Further Work

I am interested in developing a systematic partitioning scheme that would have the same efficiency using a left or right sum as using an equal partitioning scheme with the trapezoid or Simpson’s rule for area and volumes of revolution. One approach I am taking involves partitioning the y axis into equal subdivisions and using the inverse image to determine the partition of the x axis. This is shown below. This method will produce finer partitions where the function is steeper, but does not take into account the distance the function is from the axis of rotation. Hence this method is better for area approximations rather than volume approximations.

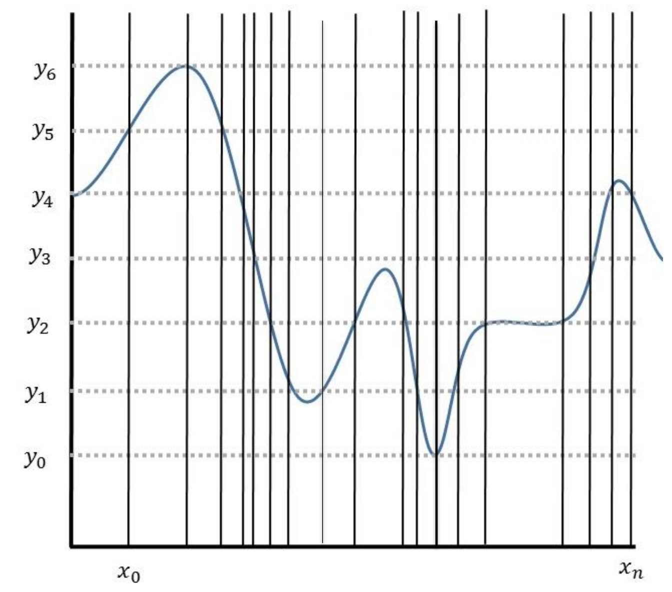

Figure IX: Partitioning the y axis and using the inverse image to determine the partition of the x axis.

Acknowledgements

I would like to thank my advisor, Professor Snipes, for her guidance in conducting this research. Thank you to Pi Mu Epsilon, the Office of the Provost, and the Kenyon Math Department for funding to present my research at Mathfest. Thank you to the Kenyon College Summer Science Scholars program for making this research possible.