Brownian Motion in the Complex Plane
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Background
Brownian motion is a model of random motion. Given a domain in the complex plane and a basepoint in the domain, start a Brownian traveler at that basepoint. The h-function of the domain gives information about where the Brownian traveler is likely to first hit the boundary of the domain. I computed the h-functions for several families of domains analytically using conformal mapping. I found instances in which non-smoothness in the boundary of a domain can be detected in the h-function. I also proved the pointwise convergence of two different sequences of h-functions. Finally, using simulations of Brownian motion, I approximated the h-functions for another family of domains. By improving the speed of the simulations, I was able to gather more data and obtain more accurate approximations of the h-functions.

Brownian Motion
Brownian motion is a model of random motion used to model the motion of molecules and fluctuations of the stock market, among other phenomena. We focus on Brownian motion in two dimensions.

h-functions
The harmonic measure distribution function, or h-function, of a domain gives information about how a Brownian particle moves in that domain. Given a domain such as the shaded region below, we start a Brownian particle moving from the basepoint z_0 until it hits the boundary of the domain. For each fixed radius r > 0, the h-function h(r) is defined to be the probability that the particle hits the boundary within distance r of the basepoint, that is, on the red portion of the boundary in the figure below.

Properties of h-functions
• The h-function is always between 0 and 1 because it represents a probability.
• The h-function h(r) is 0 for all r < r_0, where r_0 is the shortest distance from the basepoint z_0 to the boundary of the domain.
• The h-function is increasing with r. If R > r, there is a greater probability that the particle will it the boundary of the domain within distance R of z_0 than within distance r of z_0, so h(R) > h(r).
• The h-function trends towards 1 as r increases because we are guaranteed that the Brownian particle will eventually hit the boundary of the domain.

Computing h-functions Exactly
If there exists a conformal (angle-preserving) map from a domain to the interior of the unit disk, then we can compute the h-function exactly. In the example below, f is a conformal map from the domain on the left to the unit disk.

The probability that a Brownian particle starting at z_0 will first hit the red portion of the boundary in the domain on the left is equal to the probability that a Brownian particle starting at f(z_0) will first hit the red portion of the boundary in the domain on the right. This probability is given by \( \frac{d}{2\pi} \).

Convergence of h-functions
• I constructed a sequence of exterior of wedge domains with their angles increasing toward \( \pi \). I also constructed a sequence of wedge with spike domains with their spike lengths increasing toward infinity.
• The domains in both sequences look more and more like the slit plane.
• I proved that the sequences of h-functions for both of these sequences of domains converge pointwise to the h-function for the slit plane. That is, the h-functions get closer to the h-function for the slit plane.
• For the exterior of wedge domains, I extended this result to uniform convergence on any interval [r*, \( \alpha \)] with \( r^* > 0 \).

Simulations
The Cantor set is built in stages. Starting with the interval [0,1], the middle third of the interval, \( \left( \frac{1}{3}, \frac{2}{3} \right) \), is removed, leaving the intervals \([0, \frac{1}{3}]\) and \(\left[ \frac{2}{3}, 1 \right]\).

Next the middle thirds of these two intervals are removed, and so on.

Constructing the Cantor Set

The Cantor domain is the complex plane without the Cantor set, with basepoint z_0 = -1, so its boundary consists of just the Cantor set. It can be approximated in stages by using stages of the Cantor set. I ran simulations of Brownian motion in Matlab to approximate the h-functions for the first several stages of the Cantor domain.

Conjectures
• For the interior of wedge domain with angle \( \alpha < \frac{\pi}{2} \), the n-th derivative \( h^{(n)}(r) \) is continuous at \( r = 1 \) if \( 0 < \alpha < \frac{\pi}{4n} \) but discontinuous if \( \alpha \geq \frac{\pi}{4n} \).
• For the wedge with spike domain, \( h^{(n)}(r) \) is discontinuous at \( r = b + 1 \).
• These conjectures imply that corners in a domain can be detected in discontinuities in some derivative of the h-function.
• Whenever a sequence of h-functions converges pointwise to an h-function, the convergence is uniform.
• The height of the middle step of the h-function for each stage of the Cantor domain decreases monotonically as the stage increases. This could be useful in finding the exact value of the step height.

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References