Generalizing the Four Numbers Problem to Planar Graphs

Sam Justice '15 and Carol Schumacher
Department of Mathematics and Statistics, Kenyon College, Gambier, OH

Abstract

After giving a brief overview of the Four Numbers Game, we generalize it to planar graphs. In our generalization, the steps of the game alternate between the graph and its planar dual. To illustrate our generalized game, we present a case study of the game involving the self-dual 5-Wheel graph. We also discuss results for a couple of other self-dual planar graphs, and conclude with some suggestions for avenues of further exploration.

The Four Numbers Game

The Four Numbers Game begins with a square that has nonnegative integers placed on its corners. From the start square, a new square is formed whose corners lie on the midpoints of the sides of the original square. We obtain a nonnegative integer for each corner of the new square by taking the absolute difference between the two numbers on the corners of the original square that the new corner sits between. For example, if two numbers on the corners of the original square are 8 and 5, then the number on the corner of the new square that sits between the corners that have 8 and 5 on them will be $|8 - 5| = 3$.

We repeat this process of obtaining new squares until we reach a square with only zeroes on its corners. At this point, we say that the game ends since each successive square will also have zeroes on every corner. Figure 1 shows an example of a particular Four Numbers Game. We define the length of a Four Numbers Game to be the number of steps it takes to obtain a square of all zeroes, where the start square counts as step zero. With this definition of length in place, we can now state the following useful facts that have already been established about the Four Numbers Game:

- Every Four Numbers Game played with nonnegative integers has finite length.
- For every positive integer n, there exists a Four Numbers Game of length n.
- For all $k > 2$, every k-Numbers Game has finite length if and only if k is a positive power of two.

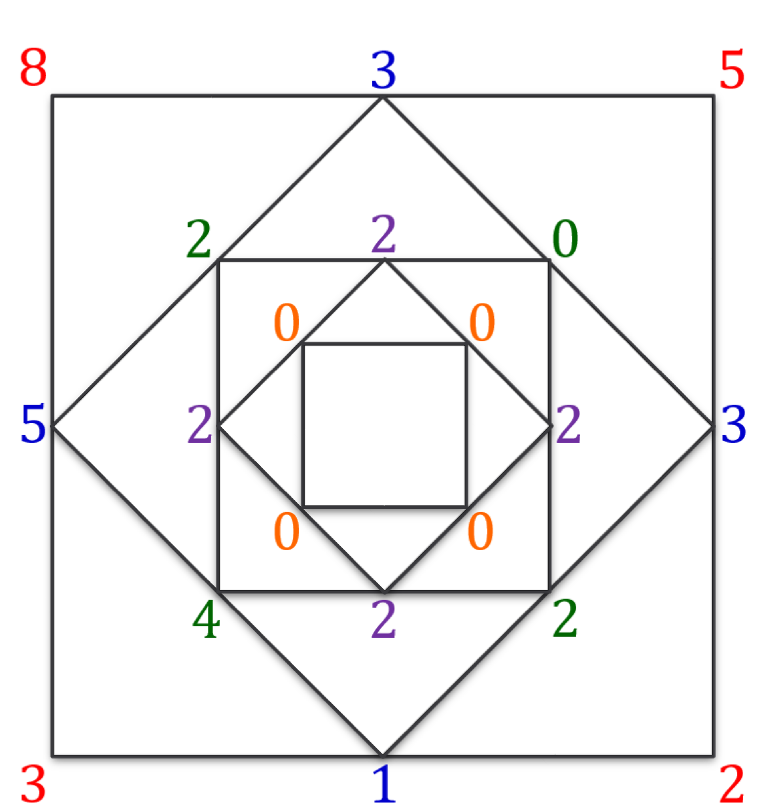

Figure 1: A Four Numbers Game of length four

Finding a Generalization

Our research goal was to generalize the Four Numbers Game to graphs. While we were not able to find a generalization that behaved well on any arbitrary graph, we did propose a game that can be played on all planar graphs. In our proposed game, successive steps go back and forth between the graph and its planar dual. We paid particular attention to games involving self-dual planar graphs. All steps of such games take place on identical graphs.

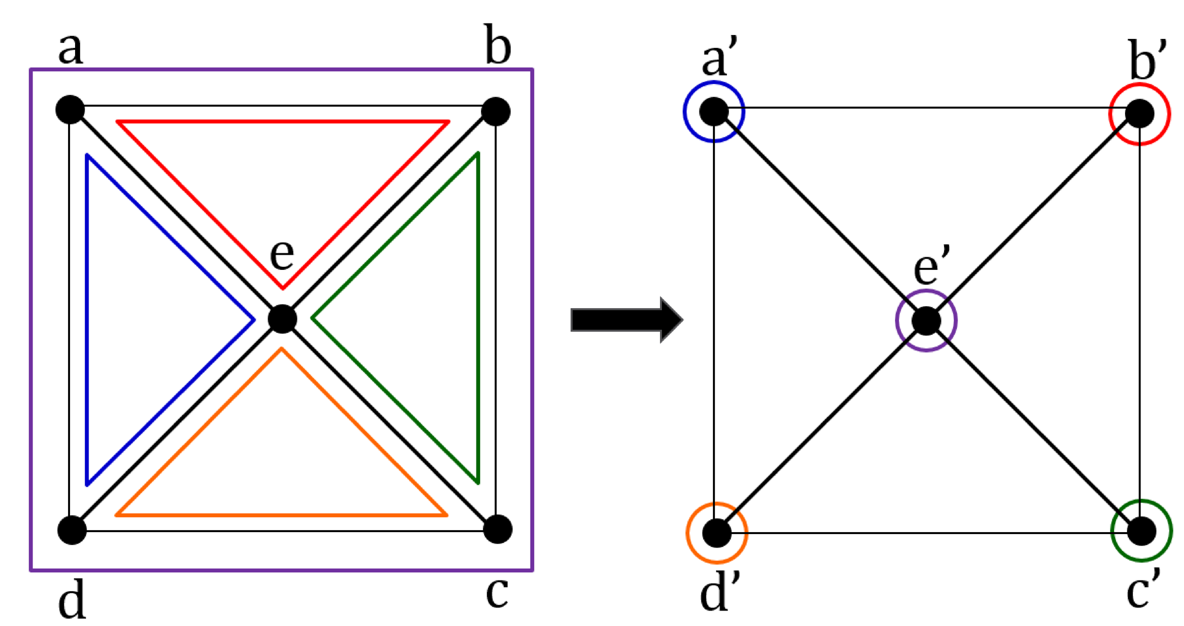

Figure 2: Obtaining each successive step of the 5-Wheel Game

Case Study: The 5-Wheel Game

It is easiest to understand our generalized game by looking at how it is played on a particular graph. Much of our research focused on the self-dual wheel graphs, and on the 5-wheel graph in particular.

We play the 5-Wheel Game as follows. First we take a 5-wheel graph and place a nonnegative integer on each of its vertices. This will serve as our start graph. To obtain the first step of the game, we associate each face of the start graph with a vertex on a new 5-wheel graph. As Figure 2 shows, we associate each triangular face with a corner vertex, and we associate the outer face with the center vertex. Every vertex of the new 5-wheel receives a nonnegative integer that is equal to the maximum pairwise difference between the numbers on the vertices of the associated face of the start graph. In Figure 2, for example, we have

$$a' = \max\{|a - d|, |a - e|, |d - e|\}.$$

We perform this process over again to obtain each successive step of the game. Repeated computer simulation (and eventually proof) led us to conclude that there are two possible end behaviors for a game. In particular, a game can either reach a fixed point or enter a cycle. Analogous to a square of all zeroes in the Four Numbers Game, a fixed point is a graph whose next step is itself. A cycle, on the other hand, is a graph that returns to itself after two or more steps of the game. We found that every cycle in the 5-Wheel Game is an 8-cycle. Figure 3 shows the general forms of fixed points and cycles, respectively.

The Length of a 5-Wheel Game

Naturally, we define the length of a 5-Wheel Game to be the number of steps it takes to for the game to either reach a fixed point or enter a cycle. Figure 4 shows a particular 5-Wheel Game that ends in a fixed point in three steps.

We were able to prove in our research that every 5-Wheel Game played with nonnegative integers has length at most five. The key to this proof is recognizing that the end behavior of an arbitrary 5-Wheel Game played with nonnegative integers a, b, c, d, and e is determined largely by the placement of the largest and smallest of these five integers on the vertices of the graph. Once we create cases according to where the largest and smallest of a, b, c, d, and e are positioned, it is a matter of simple computation to observe that the game ends in five steps or fewer in each case. We were able to make these computations less tedious by recognizing that there are certain game configurations that all start graphs tend to reach after a step or two.

Further Results

We analyzed games played on other graphs besides the 5-wheel, and in some cases we were able to draw nice conclusions about the lengths of such games. In particular, we proved the following two results:

- Every 4-Wheel Game played with nonnegative integers has length at most three.
- For all $n \geq 2$, every n-leafed Dipole Rose Game played with nonnegative integers has length at most three.

Figure 5 shows generic start graphs for the 4-Wheel and 3-Leafed Dipole Rose Games. These games were relatively simple to analyze. We proved the upper bound on the length of the 4-Wheel Game through case-by-case analysis based on the relative size of the numbers on the start graph, and we proved the upper bound on the length of the n-leafed Dipole Rose Game by direct computation.

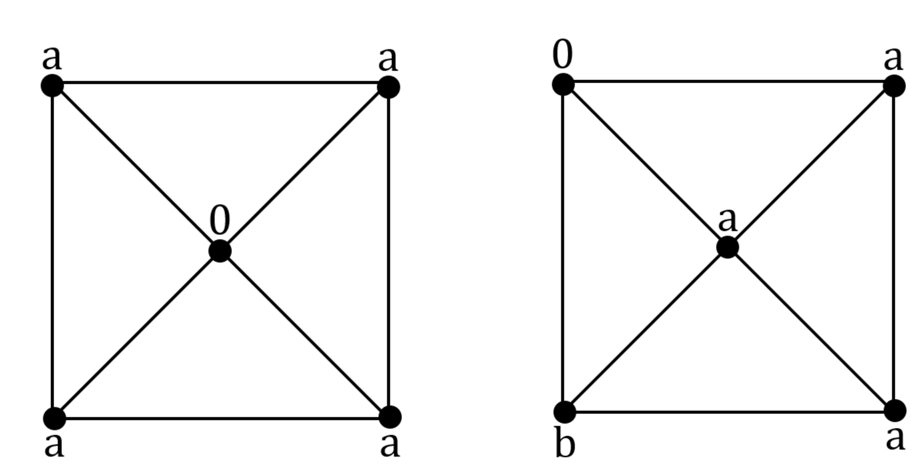

Figure 3: A general fixed point (left) and cycle (right) of the 5-Wheel Game

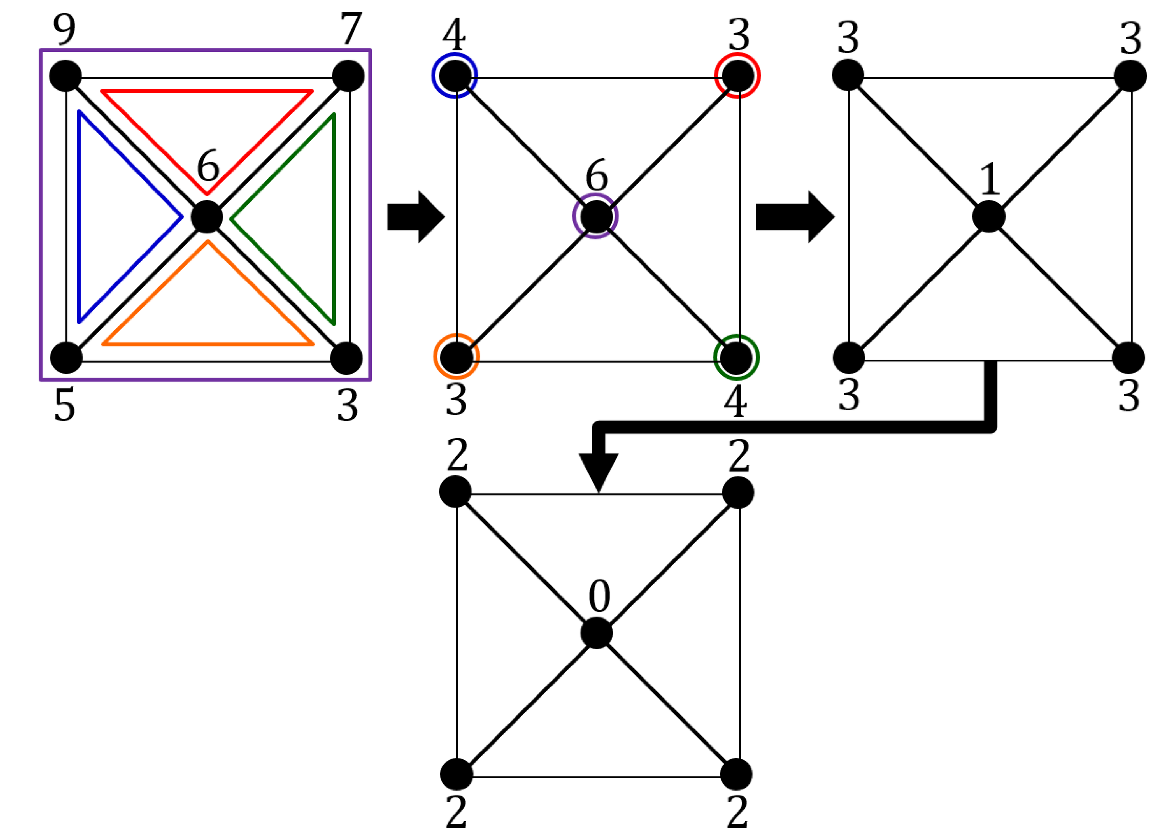

Figure 4: A 5-Wheel Game of length three

Frontiers

Our research creates plenty of avenues for further exploration, as there are an infinite number of planar graphs that we can play our generalized game on. Future research might focus more on games played on graphs that are not self-dual, and whether we can find any general patterns for the lengths of such games. Also, we still have much to do in order to completely understand games played on the wheel graphs. One question of interest is whether there is an upper bound on the length of an n-Wheel Game. That is, if we keep increasing the order of the wheel that we play on, does the maximum length of the game plateau or increase without bound? This is not an easy question to answer since the games become exceedingly complex to analyze as the order of the wheel grows.

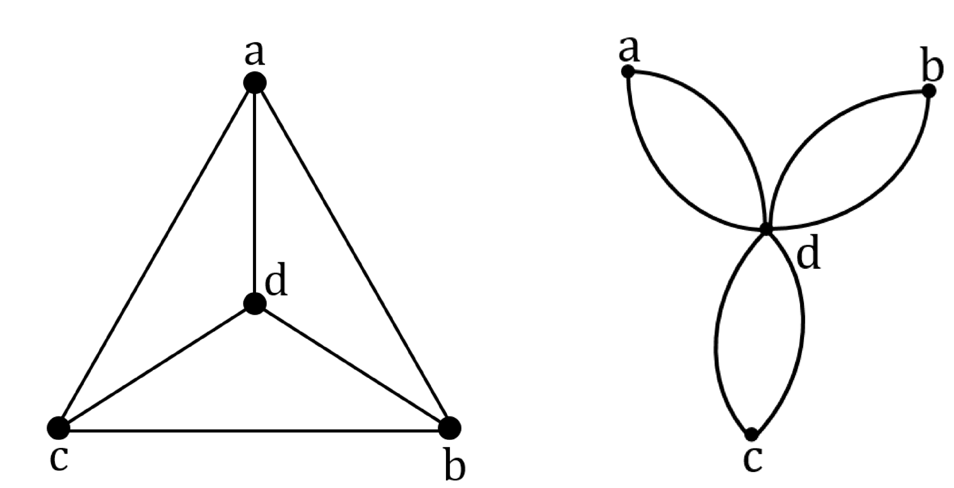

Figure 5: A generic 4-Wheel Game (left) and 3-Leafed Dipole Rose Game (right)

Acknowledgments

I would like to thank Professor Carol Schumacher for her guidance and enthusiasm over the course of this project, as well as the Kenyon Summer Science Scholars program for giving me this opportunity. Additionally, I would like to thank the Mathematical Association of America for allowing me to present this research at MathFest.

References

1. J.D. Sally and P.J. Sally, *Roots to Research: A Vertical Development of Mathematical Problems*. American Mathematical Society, Providence, RI, 2000.