Axiomatic Informational Thermodynamics
Austin Hulse
Kenyon College

Abstract
Axiomatic approaches to thermodynamics build the theory up from minimal assumptions, elucidating its basic concepts and logical structure so as to understand the kinds of systems to which it applies. We adapted the axiomatic system of R. Giles (Mathematical Foundations of Thermodynamics, Pergamon Press, 1964) to include a description of the acquisition and use of information by thermodynamic agents. Our new axioms govern the creation and erasure of physical records and their ability to store information. From our axioms, we are able to establish several important results, including the connection between thermodynamic entropy (which determines the irreversibility of a possible process) and the Shannon-Hartley measure of the information contained in an agent’s memory. In our structure, systems are regarded as sets of distinct states, of which each state is a possible true state of the system. The Second Law of Thermodynamics takes a very simple form: the impossibility of a process taking a system to a proper subset of itself.

Processes
All processes involve the help of some arbitrary apparatus, such as a catalyst, which does not change in state.

\textbf{Nondeterministic processes} \(a \rightarrow b\):
- This means \(a\) evolving into \(b\) is possible

\textbf{Processes between sets} \(A \rightarrow B\):
- For all \(a \in A\) there exists a \(b \in B\) such that \(a \rightarrow b\)
- For all \(b \in B\) there exists an \(a \in A\) such that \(a \rightarrow b\)
- For all \(a \in A\) and any state \(c\), \(a \rightarrow c\) implies \(c \in B\)

\textbf{Notation} \(G = (A, B)\):
- \(G\) is reversible if \(A \rightarrow B\) and \(B \rightarrow A\)
- \(G\) is natural irreversible if \(A \rightarrow B\) and \(B \nrightarrow A\)
- \(G\) is antinatural irreversible if \(B \rightarrow A\) and \(A \nrightarrow B\)
- \(G\) is impossible if \(A \rightarrow B\) and \(B \nrightarrow A\)

\textbf{Algebra of processes} \(G = (A, B)\) and \(D = (C, D)\):
- \(G + D = (A + C, B + D)\)
- \(\neg D = (D, C)\) which implies \(G \rightarrow D = (A + D, B + C)\)
- The \(0\) process is any reversible single state process
- Thus if \(A \rightarrow B\), then \(G \rightarrow 0\)

\textbf{Driving processes backwards} \(G \rightarrow D\):
- This means \(G \rightarrow D\) is natural (i.e. \(A + D \rightarrow B + C\))

\textbf{Record states} \((r, s)\):
- Axiom: For all \(a\) and \(r\), \(a \rightarrow a + r\)
- From this we find that for all \(r\) and \(s\), \(r \rightarrow s\)

\textbf{Information processes} \(G = (I, J)\):
- Axiom: \(\#I \leq \#J\) implies \(I \rightarrow J\) and \(G \rightarrow 0\)
- Thus for all \(I\) and \(J\), either \(I \rightarrow J\) or \(J \rightarrow I\) or both

Comparison Hypothesis
Both axiomatic formulations that we looked at had the following as either an axiom or theorem:
- \(a \rightarrow b\) and \(a \rightarrow c\) implies \(b \rightarrow c\) or \(c \rightarrow b\)

Giles took it as an axiom, but Lieb and Yngvason tried to find alternative axioms from which the comparison hypothesis would be derived. In each case, the hypothesis was required for proving the existence of entropy.

In our formulation, the comparison hypothesis applies only to information states, not all states. But from our axioms about information states, we can measure entropy by comparing the amount of information required to make the process occur.

Demonics
The above graphic of the demon shows how information can be created from thermal fluctuations.

\textbf{Symbolically}:
- \(a + r \rightarrow \{a_0, a_1\} + r\)
  - The demon inserts a partition into the box
- \(\{a_0, a_1\} + r \rightarrow \{a_0 + r_0, a_1 + r_1\}\)
  - The demon learns the location of the particle
- \(\{a_0 + r_0, a_1 + r_1\} \rightarrow a + \{r_0, r_1\}\)
  - The demon removes the partition, retaining its memory

Demons are also given the ability to reverse natural processes starting with a single state by ‘learning’ and creating a large information state. Since one does not know what one will eventually learn, we introduced nondeterministic processes.

Irreversibility and Entropy
These functions are only defined for information states/processes and single states/single state processes.

\textbf{The Second Law}:
- If \(B\) is a proper subset of \(A\), then \(A \nrightarrow B\)

\textbf{Irreversibility}:
- \(I(G + D) = I(G) + I(D)\)
- \(I(G) > 0\) if and only if \(G\) is natural irreversible
- \(I(G) = 0\) if and only if \(G\) is reversible
- \(I(G) < 0\) if and only if \(G\) is antinatural irreversible
- For information processes \((I, J)\), \(I(G) = \log \#I / \#J\)

\textbf{Entropy}:
- \(S(A + B) = S(A) + S(B)\)
- \(S(A) = S(B)\) if and only if \((A, B)\) is reversible
- \(S(A) < S(B)\) if and only if \((A, B)\) is natural irreversible
- For information states \(I, S(I) = \log \#I\)

Our largest differences from Giles are colored blue. Most of the text in black has analogous ideas in Giles.

References