Triangular and Polygonal Triples

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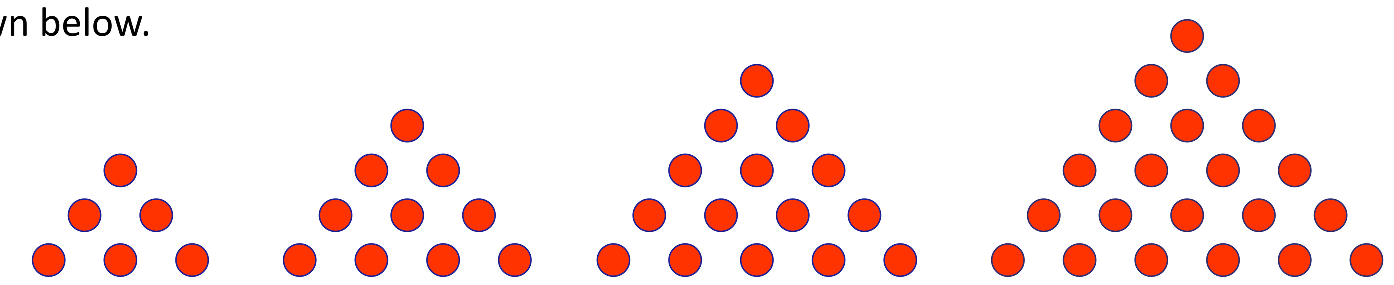
Abstract

The equation $a^2+b^2=c^2$ is one of the most famous equations in the world, due to its role in the Pythagorean Theorem. One generalization of this is the equation $a^n+b^n=c^n$, which is well known because of Fermat's Last Theorem. Recognizing that a Pythagorean triple (a,b,c) corresponds to three square numbers a^2, b^2, and c^2, the last of which is the sum of the first two, we can examine a second way of generalizing Pythagorean triples. In particular we consider the question "when is the sum of two triangular numbers a triangular number?" or more generally, "when is the sum of two polygonal numbers a polygonal number?" The answer is found parametrically, by finding polygonal triples of the form $(n,x,n+k)$, where x and n can be calculated given a value for k. The triangular case will be covered in detail, and examples of the general polygonal solution will be given.

Triangular Numbers

A triangular number is a natural number that can be put into the shape of an equilateral triangle. The n^{th} triangular number is denoted by T_n. Examples of triangular numbers are shown below.



$T_3=6$ $T_4=10$ $T_5=15$ $T_6=21$

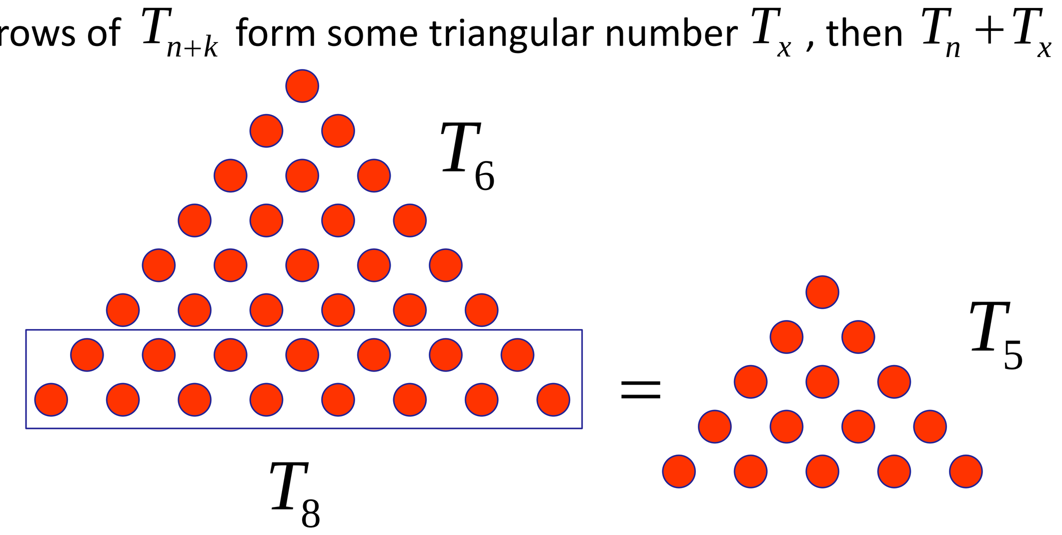
Each triangular number can be constructed by adding a row to the previous triangular number. For example, $T_5=T_4+5$. Each successive row is one piece longer than the previous row, suggesting the formula

$$T_n=1+2+3+\ldots+n=\frac{n(n+1)}{2}$$

Triangular Triples

- A positive integer triple (a,b,c) is a triangular triple if $T_a+T_b=T_c$.
- Example: $T_3+T_5=6+15=21=T_6$

•Example: $T_6+T_5=T_8=T_{6+2}$. This is because the bottom two rows of T_8 form T_5. So if the bottom k rows of T_{n+k} form some triangular number T_x, then $T_n+T_x=T_{n+k}$.



Assume $T_n+T_x=T_{n+k}$, where $x,k\in N$. Then $n(n+1)+x(x+1)=(n+k)(n+k+1)$, so $x(x+1)=2nk+k(k+1)$. Therefore n will be an integer exactly when $x(x+1)\equiv k(k+1)\bmod(2k)$. Since we need n to be a positive integer for *(n, x, n+k)* to be a triangular triple, we can use this congruence to find values of x and k that force n to be a positive integer when $T_n+T_x=T_{n+k}$. This congruence can be solved by fixing k and using the prime factorization of k. Theorem 1 provides a complete description of triangular triples with odd k.

Theorem 1

Let k be an odd positive integer with prime factorization $k=\prod_{i=1}^{s}p_i^{r_i}$. Let $n=\frac{x(x+1)-k(k+1)}{2k}=\frac{T_x-T_k}{k}$. Then *(n, x, n+k)* is a triangular triple if and only if $x>k$ and $x\equiv 0$ or $-1 \pmod{p_i^{r_i}}$ for $1\le i\le s$.

Proof of Theorem 1

Fix k odd. Recall that n is an integer if and only if $x(x+1)\equiv k(k+1)\pmod{2k}$ so we start by solving this congruence. Since by assumption k is odd, $(k+1)/2$ is an integer, so $k(k+1)\equiv 2k\frac{(k+1)}{2}\equiv 0\pmod{2k}$. Because 2 and k are relatively prime, $x(x+1)\equiv 0\pmod{2k}$ if and only if $x(x+1)\equiv 0\pmod 2$ and $x(x+1)\equiv 0 \bmod(k)$. One of x, $x+1$ is always even, so the former congruence is always true. To solve the latter congruence, note that each prime power factor of k is relatively prime, so $x(x+1)\equiv 0\pmod k$ if and only if $x(x+1)\equiv 0\pmod{p_i^{r_i}}$ for $1\le i\le s$. Because x and $x+1$ are relatively prime, each congruence $x(x+1)\equiv 0\pmod{p_i^{r_i}}$ has only the solutions $x\equiv 0\pmod{p_i^{r_i}}$ and $x\equiv -1\pmod{p_i^{r_i}}$. Therefore $x(x+1)\equiv 0\pmod{2k}$ and n is an integer if and only if $x\equiv 0$ or $-1\pmod{p_i^{r_i}}$ for $1\le i\le s$. But if $x\le k$, then n is not positive, so for *(n, x, n+k)* to be a triangular triple we require the additional constraint that $x>k$. Note that $T_n+T_x=T_{n+k}$ by construction, so *(n, x, n+k)* is in fact a triangular triple.

Theorem 2

Let k be an even positive integer with prime factorization $k=2^t\prod_{i=1}^{s}p_i^{r_i}$. Let $n=\frac{x(x+1)-k(k+1)}{2k}=\frac{T_x-T_k}{k}$. Then *(n, x, n+k)* is a triangular triple if and only if $x>k$, $x\equiv 2^t$ or $2^t-1\pmod{2^{t+1}}$, and $x\equiv 0$ or $-1\pmod{p_i^{r_i}}$ for $1\le i\le s$.

Example for Odd k

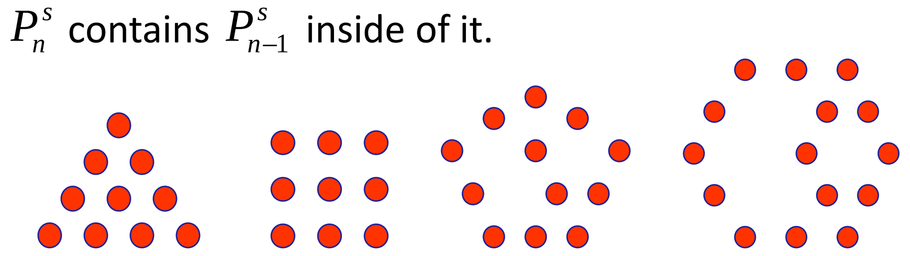
Suppose that $k=45=3^2\cdot 5$. Then to generate a triangular triple, we find some x so that $x\equiv 0$ or $-1\pmod 9$ and $x\equiv 0$ or $-1\pmod 5$. Suppose we pick $x\equiv -1\pmod 9$ and $x\equiv 0\pmod 5$. Using the Chinese Remainder Theorem or just by guessing, we see that $x\equiv 35\pmod{45}$ is the general solution to this system of two congruences. Since we need $x>k$, one valid choice is $x=80$. Calculating n as in Theorem 1, we obtain $n=49$. Therefore (49, 80, 94) is a triangular triple.

Example for Even k

For an example using an even k, let $k=600=2^3\cdot 3\cdot 5^2$. Then by Theorem 2, we must find some x satisfying $x\equiv 0$ or $-1\pmod 3$, $x\equiv 0$ or $-1\pmod{25}$, and $x\equiv 8$ or $7\pmod{16}$. Suppose we choose $x\equiv 0\pmod 3$, $x\equiv -1\pmod{25}$, and $x\equiv 7\pmod{16}$. Again, the Chinese Remainder Theorem can be used to determine that the solution to this system of congruences is $x\equiv 999\pmod{1200}$. Since $999>600$ we can use this as our x. Then we can calculate n as in the statement of Theorem 2, giving $n=532$. Therefore (532, 999, 1132) is a triangular triple.

Polygonal Numbers

Polygonal numbers are numbers that can be represented as a regular polygon. The n^{th} polygonal number of s sides is $P_n^s=\frac{n((s-2)n-(s-4))}{2}$, or equivalently the sum of an arithmetic series of n terms with first term 1 and common difference $s-2$. Examples of polygonal numbers are shown below. Notice that for any $s>2$, the shape P_n^s contains P_{n-1}^s inside of it.



P_4^3 P_3^4 P_3^5 P_3^6

Polygonal Triples

A positive integer triple (a,b,c) is a polygonal triple if for some integer $s>2$, $P_a^s+P_b^s=P_c^s$. The methods used to find triangular triples were generalized and used to find polygonal triples, so polygonal triples were found in the form *(n, x, n+k)*. The full solution depends on the common factors of $s-2$ and k, as well as of $s-4$ and k. The solution in the simplest case is stated below. Notice that when s=3, this result simplifies to Theorem 1.

Theorem 3

Let k be odd with prime factorization $k=\prod_{i=1}^{t}p_i^{r_i}$. Let $s>2$ be an integer and assume $\gcd(s-2,k)=\gcd(s-4,k)=1$. Let $n=\frac{x(x+1)-k(k+1)}{2k}=\frac{T_x-T_k}{k}$. Then *(n, x, n+k)* is a polygonal triple for polygons with s sides if and only if $x>k, x\equiv k\pmod{s-2}$, and $x\equiv 0$ or $1-2(s-2)^{-1}\pmod{p_i^{r_i}}$ for $1\le i\le t$.

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