Unraveling Entanglement: Correlation and Sharability within Quantum Mechanics and Beyond

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Abstract: The mysterious phenomenon of quantum entanglement has puzzled physicists for decades. Two entangled particles behave in highly correlated ways -- an almost "telepathic" connection that can extend over any distance. In quantum mechanics, no third particle is able to tap into this link and share in the entanglement relationship. This fact is called monogamy of quantum entanglement. In this talk, we will explore weaker degrees of entanglement that may be shared among up to three, four, or N particles.

The ideas of sharability and monogamy are clearer if we approach them in a more general way, using a simple framework that describes any sort of correlated behavior, including that of entangled quantum particles. The monogamy of quantum entanglement does not depend on the detailed mathematics of quantum theory, but can be deduced directly from the observed correlations of pairs of particles.

Introduction: We will explore the principle of Sharability within quantum mechanics and then in a more general framework. Sharability is a type property of any state that describes two subsystems.

Within quantum mechanics, we describe mixed states of systems using density operators. These abstract entities are realized by understanding them to be spin-1/2 systems.

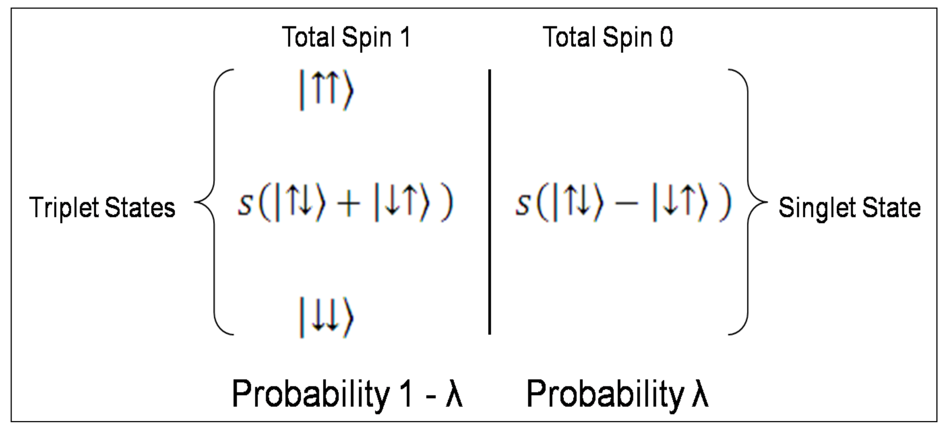
Within general sharability, we consider systems made of several binary observables. The example of binary observables we use to explain general sharability is "yes or no questions".

Quantum Sharability

Quantum Sharability: A two-system quantum state Ψ is n-m sharable if and only if there exists a density operator ρ describing the entourage of (n+m) sub-systems such that reduction of ρ to two sub-systems always results in Ψ.

Results: Sharability of Werner States is an interesting and significant property because the set of Werner States can be divided into sharability classes. These classes range from including states which are not sharable at all, as in the singlet state, to including states which are arbitrarily sharable, as in the completely mixed identity operator.

Method: We considered sharability of only the simplest, most symmetric quantum states, namely, Werner states. These are the rotationally symmetric two qubit density operators. We can write the Werner states in a basis of triplet and singlet states:

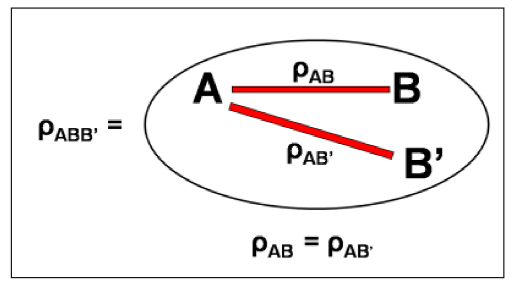


Here is the form of the general Werner State:

$$\rho_\lambda = \lambda|\Omega\rangle\langle\Omega| + (1-\lambda)\left(\frac{1}{3}|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + \frac{1}{3}|\downarrow\downarrow\rangle\langle\downarrow\downarrow| + \frac{1}{3}s^2(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)(\langle\uparrow\downarrow| + \langle\downarrow\uparrow|)\right)$$

Each Werner state is uniquely characterized by its value of λ.

A Werner state is sharable if there exists a density operator whose A-B subsystems are described by that Werner state.



ρ_{AB} is then said to be 1-2 sharable

The A-B subsystems of a whole-system density operator ρ will be maximally sharable when each subsystem is as close as possible to the singlet state while ρ is a true density operator.

A measure of closeness of each A-B subsystem to the singlet state is a projection, H, of ρ into each singlet state.

$$H_{mn} = \sum_{j,k=1}^{m,n} |\Omega^{jk}\rangle\langle\Omega^{jk}| \otimes \mathbb{I}^{\text{rest}}$$

Two useful corollaries we proved:

•The pure state which has the greatest expectation of H will be the eigenstate(s) with largest eigenvalue. We call this space the *ceiling space*.
•The density operator which maximizes the expectation of H is an even mixture of pure states in the ceiling space.

With these, we find a maximally m-n sharable Werner state by finding the largest eigenvalue of H_{mn}.

Considering our systems to be spin-1/2 particles, we rewrite H as:

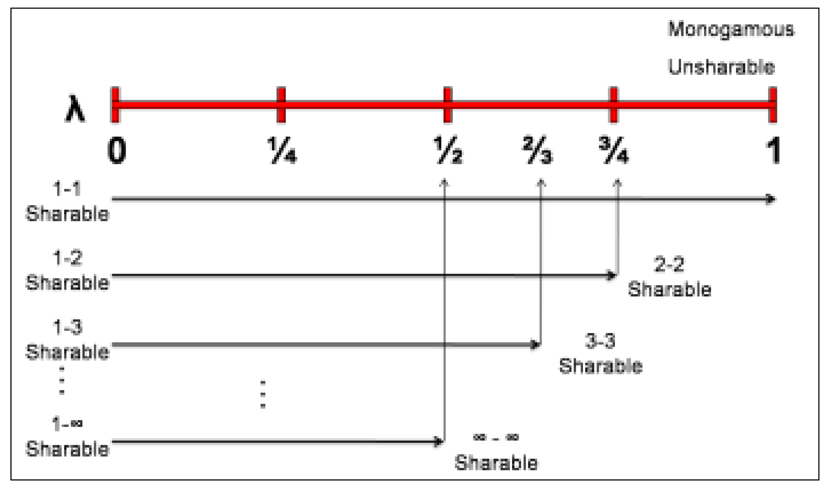
$$H_{mn} = \sum_{j,k=1}^{m,n} \frac{1}{4}\mathbb{I} - \vec{S_j}\cdot\vec{S_k}$$

In quantum mechanics, the eigenvalues of the total spin operator are:

$$S^2|s,m\rangle = s(s+1)|s,m\rangle$$

We then rewrite H_{mn} in terms of total spin operators to find that the largest eigenvalue for a given H_{mn} is,

$$\Lambda = \frac{mn}{2} + \frac{m}{2} \quad \text{and so} \quad \lambda = \frac{\Lambda}{mn} = \frac{n+1}{2n}.$$



General Sharability

General Sharability: A two-system general state Ψ is n-m sharable if and only if there exists a set of joint probabilities, obeying rules of probability and locality.

Results: Sharability of General States is an interesting and significant property because it places restrictions on what statistical relationships are possible of systems in general. The beauty of this result is that these restrictions are based only on our general framework and locality.

General Framework:

•**Outcome:** result of an observation, distinguishable from other results (e.g. +, –,a,b}

•**Observable:** set of outcomes (e.g. A = {+, –})

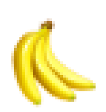
•**System:** set of observables
(e.g. S = {A,B,C} = {{$+_A$, $-_A$},{$+_B$, $-_B$},{$+_C$, $-_C$}})

•**State:** set of probability distributions, with one distribution for each joint observable (example below)

Method: We considered the following type of systems:

S_1 = {A_1,B_1}, S_2 = {A_2,B_2}, where A,B = {+, –}

and considered states where A-A joint observables always agree, A-B or B-A joint observables agree with probability p, B-B joint observables agree with probability q, and any single observation returns + or – with equal probability.

P(agree)		
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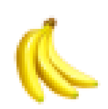
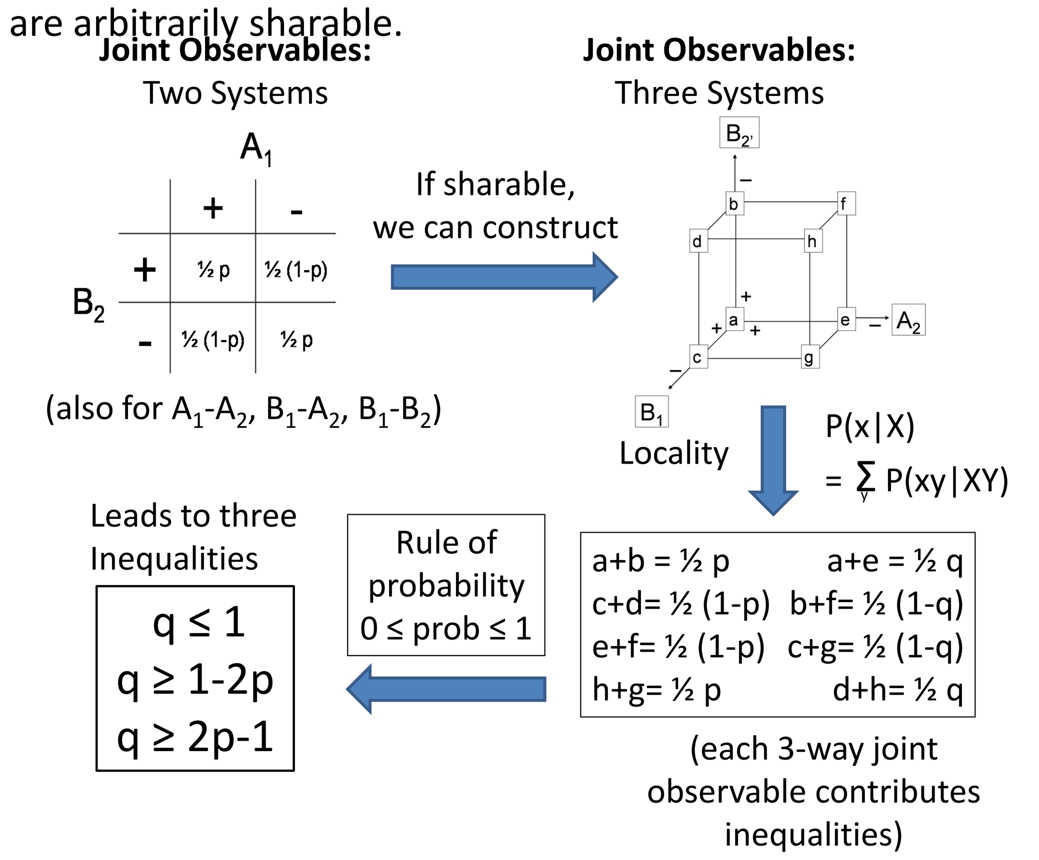
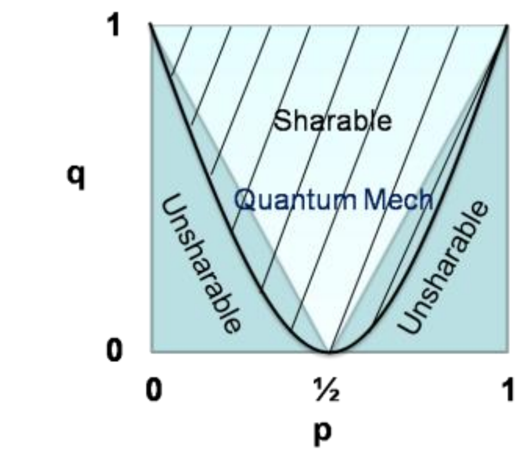
	p	q

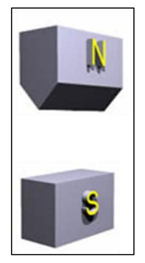
We show that certain p-q states are unsharable, while others are arbitrarily sharable.



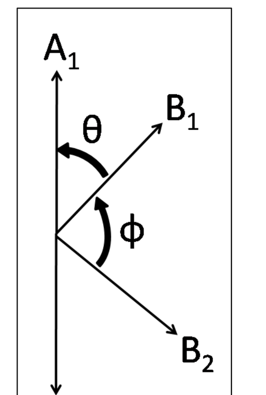
We represent these inequalities with the following diagram:



We consider a real-world example and show that certain unsharable relationships are expressed in nature.



Stern-Gerlach



QM gives us probabilities of agreement:
P(agree) = $\sin^2(\chi/2)$
P(A-A agree) = $\sin^2(\pi/2) = 1$
P(A-B agree) = P(A-B agree) = $\sin^2(\theta/2) = p$
P(B-B agree) = $\sin^2(\phi/2) = \sin^2(\theta - \pi/2) = q$

Using trig identities, we can write q in terms of p as:

$$q = (2p-1)^2$$

As seen above, we have shown that there exist states in nature that are unsharable, and their unsharability can be derived simply on account of their observed statistical relationships.