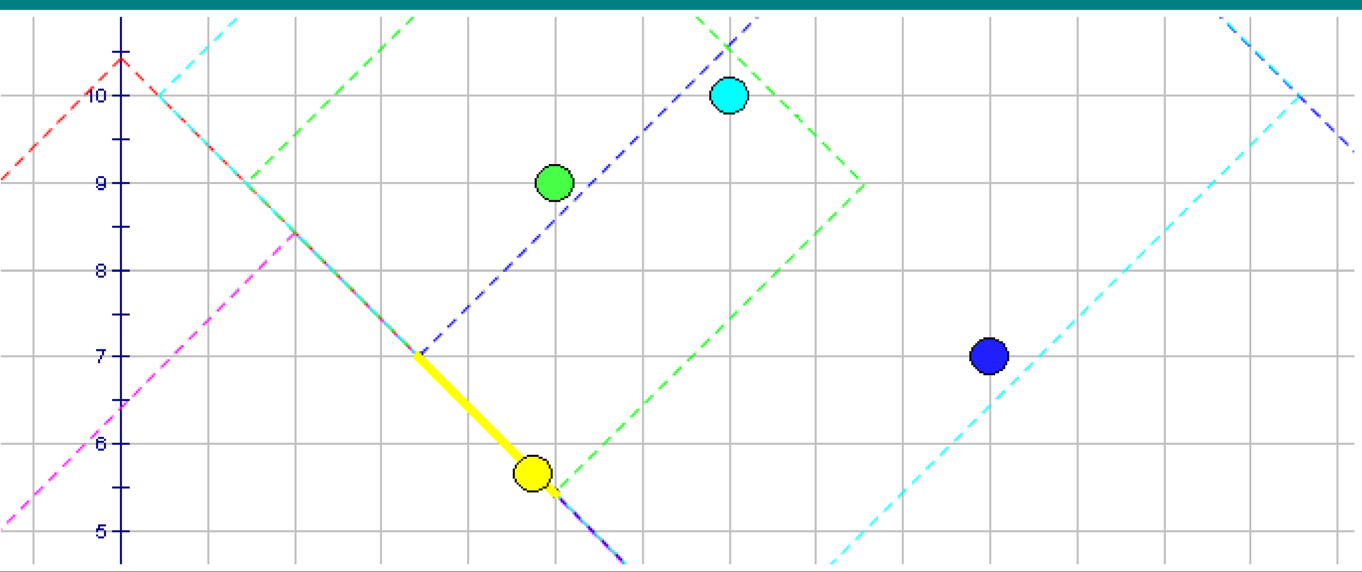


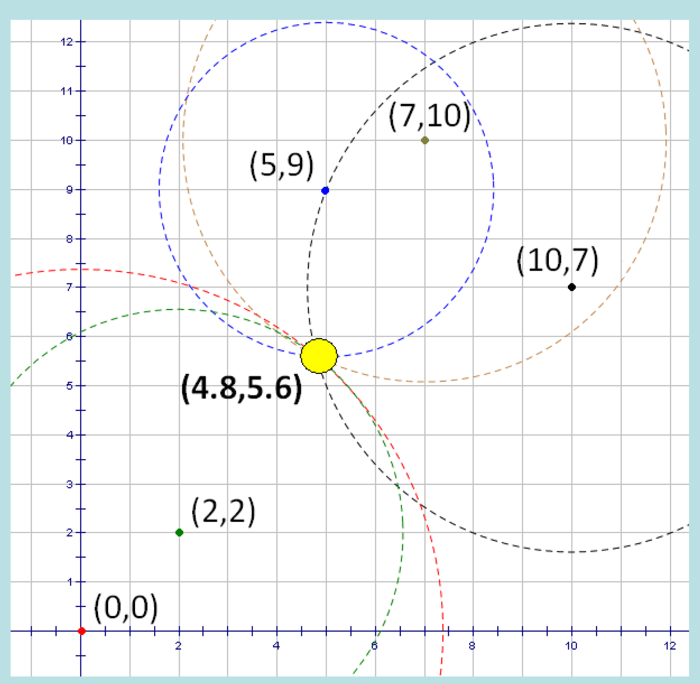
Finding the Metric Center

Hannah Ross; Prof. Carol S. Schumacher

ABSTRACT

This past summer Professor Carol Schumacher and I worked on generalizing the concept of center of mass for a finite number of points to an abstract metric space. We found that in a finite dimensional normed linear space, any finite set will have at least one metric center. Unlike the center of mass of a finite set in Euclidean space, the metric center of a set in a general normed linear space need not be unique, and all of the metric centers of a given set K together will form a convex set. This set of metric centers is preserved by isometries. Another property of the metric centers is that, if we have two metric centers x and y for the same finite set K, the distance from x to k is the same as the distance from y to k for every k in K. In addition, we spent a lot of time examining metric centers under the taxicab metric. We found that, in the plane, a taxicab metric center will always be either a point or a line segment.

Ultimately, we discovered that the function used to define the metric center is a convex function. This tells us that all of the directional derivatives of the function will exist for all points in its domain. Therefore, we can always find the metric center by following the negative of the gradient down to a minimum. The convexity of the function guarantees that any local minimum will be a global minimum, so we can be sure that this method will lead us to a metric center.



MATERIALS

Maple

C++

Geometer's Sketchpad

REFERENCES

Closer and Closer

Taxicab Geometry

OBJECTIVE

To generalize the concept of center of mass to an abstract metric space.

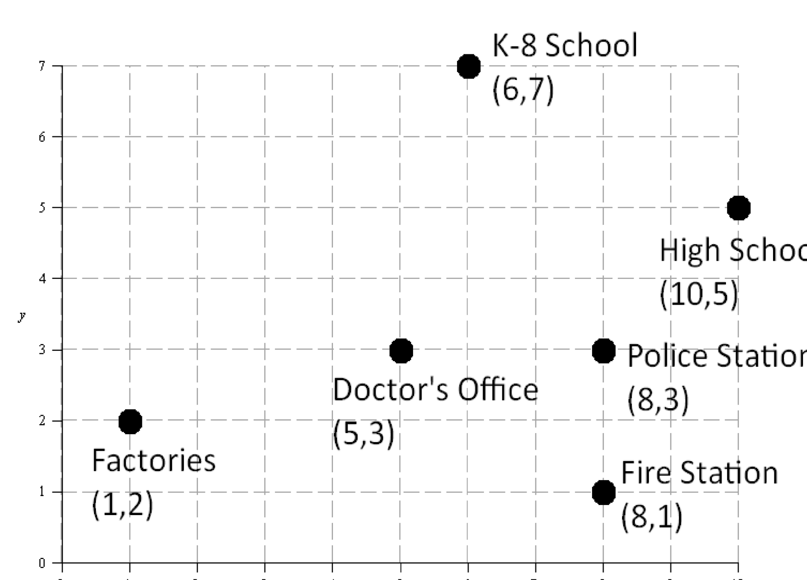
INTRODUCTION

The concept of center of mass has proved highly useful in fields across the sciences, from mathematics to astrophysics to aeronautical engineering. It is used to describe the "balancing point" of an object, or a point "as close as possible" to every element of a given set. However, as it stands, center of mass is specific to the Euclidean metric, excluding the many other metrics and metrics spaces that appear throughout mathematics.

Therefore, this summer, Professor Schumacher and I worked to generalize the concept of center of mass to an abstract metric space. Would the metric center, as we chose to call it, always exist? Would it always be unique like the center of mass under the Euclidean metric? How could we characterize and locate the metric center in any give metric space?

We spent a lot of time in particular with the taxicab metric, a distance formula specifically concerned with city distances in terms of the number of blocks between two locations. This provided a number of puzzles in its own right – we found that anything that could behave strangely tended to do so under the taxicab metric. In addition, the taxicab metric creates a good environment for a practical example of when you might want to locate the metric center of a set. Consider the mock city plotted on the graph below.

Where should we build a hospital in order to place it as close as possible to all of the locations show below?

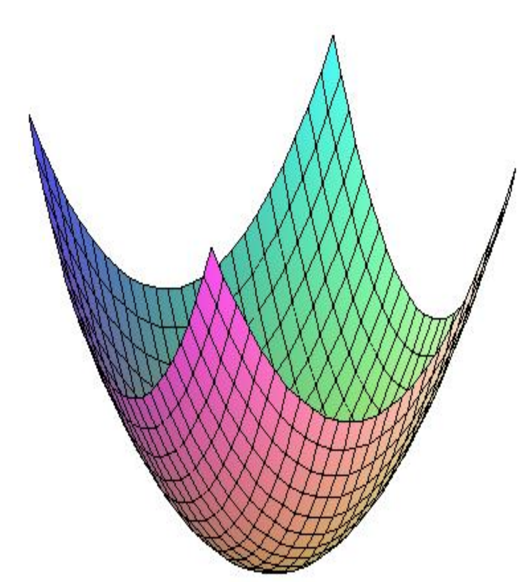


DEFINITION

Let K be a finite set in a metric space (X,d). Define a function $f: K \to R$ such that

$$f(x)=\sum_{k\in K} d(x,k)^2$$

A metric center, x_K, of K is a point at which f reaches an absolute minimum.

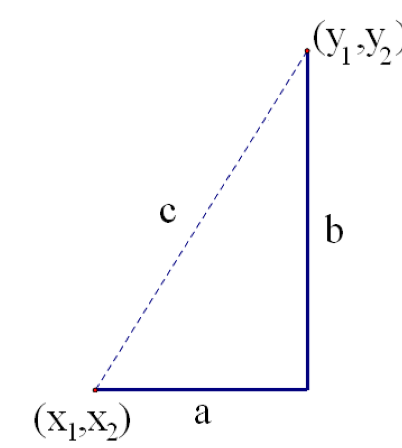


EXISTENCE

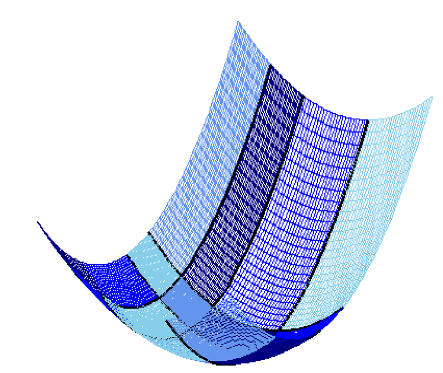
Let (X,d) be a finite dimensional normed linear space with a finite subset K. Then f will reach an absolute minimum in X; that is, the metric center for K in X must exist.

COMPARISON

Euclidean

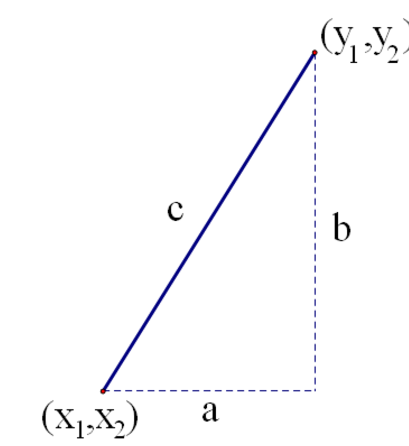


$$d(x,y)=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$$

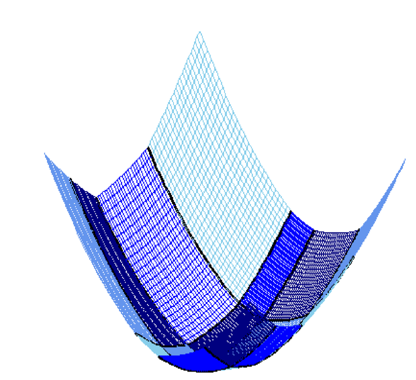


$$f_E(x)=\sum_{k\in K}(x_1-k_1)^2+(x_2-k_2)^2$$

Taxicab

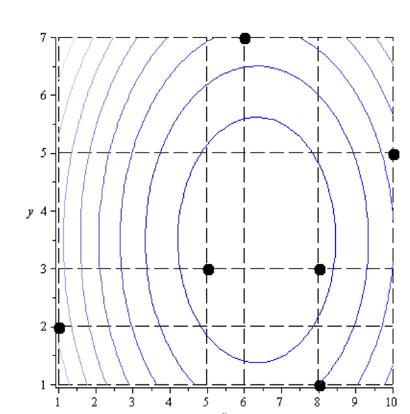


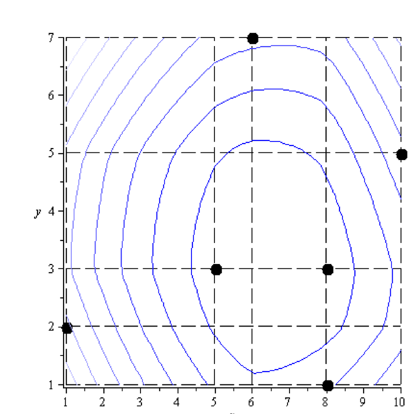
$$d(x,y)=|x_1-y_1|+|x_2-y_2|$$



$$f_T(x)=\sum_{k\in K}(|x_1-k_1|+|x_2-k_2|)^2$$

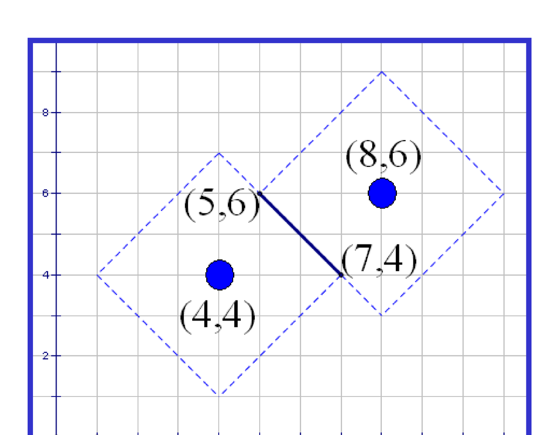
Contour Plots for f_E and f_T



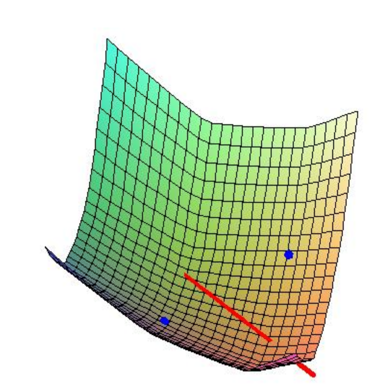


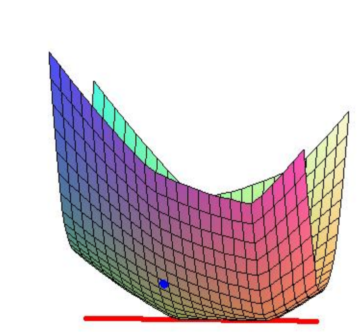
UNIQUENESS

The metric center of a set K need not be a single point; it may be that f is minimized at several points. Consider the metric center of two points under the taxicab metric:



The metric center for a set of two points. Note: f((5,6)) = f((7,4)).





The graph of f for a set of two points.

CONVEXITY

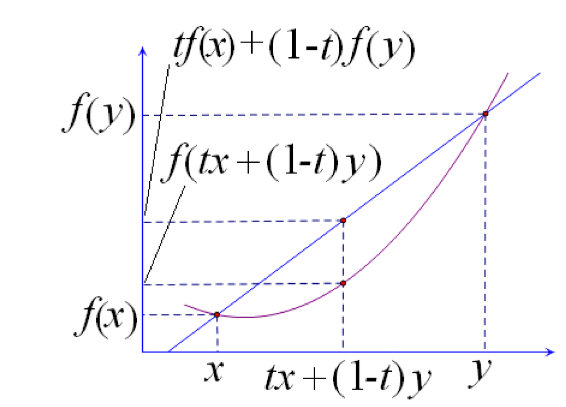
Definition

A real-valued function g is convex, provided that given any two elements of its domain, x and y, and some $t \in [0,1]$,

$$g(tx+(1-t)y)\le tg(x)+(1-t)g(y).$$

Properties of Convex Functions

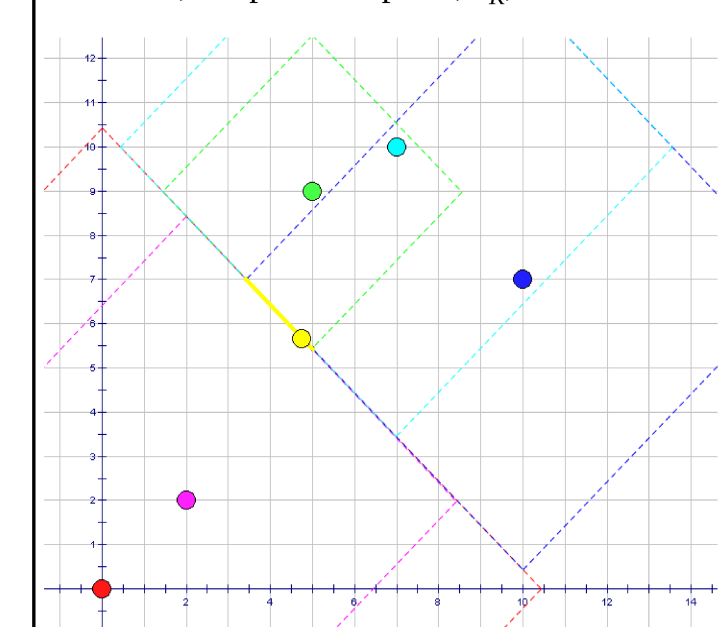
- Every local minimum is a global minimum
- The set of absolute minima is a convex set
- The directional derivatives of f exist at all points in X



ALGORITHM

To find the metric center of a given set of points:

1. Plot f for set K
2. Choose a starting point (x,y) – (e.g. a point in K)
3. Find directional derivatives at (x,y)
4. Move one step in the direction opposite the gradient to a new point (x',y')
5. Calculate $f(x',y')$
6. Repeat from Step 3 using (x',y') as long as f of the new point is less than f of the previous point
7. Then, the previous point, x_K, is a metric center for K



8. To find the rest of the metric centers, calculate $d(k,x_K)$ for all k in K
9. Draw circles with a radius of the appropriate distance, $d(k,x_K)$, around each k in K
10. The intersection of these circles is the entire set of metric centers for K

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