Brownian Motion in the Complex Plane

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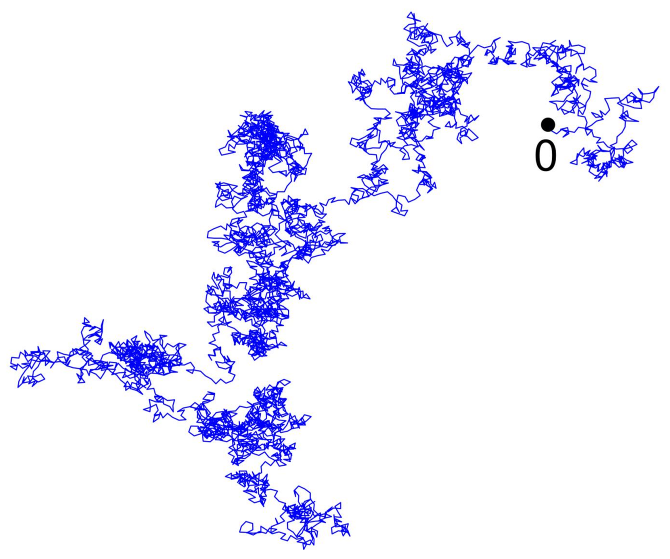
Background

Brownian motion is a model of random motion. Given a domain in the complex plane and a basepoint in the domain, start a Brownian traveler at that basepoint. The h-function of the domain gives information about where the Brownian traveler is likely to first hit the boundary of the domain. I computed the h-functions for several families of domains analytically using conformal mapping. I found instances in which non-smoothness in the boundary of a domain can be detected in the h-function. I also proved the pointwise convergence of two different sequences of h-functions. Finally, using simulations of Brownian motion, I approximated the h-functions for another family of domains. By improving the speed of the simulations, I was able to gather more data and obtain more accurate approximations of the h-functions.

Brownian Motion

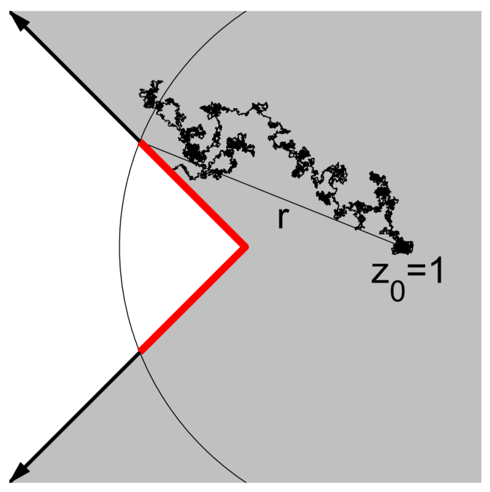
Brownian motion is a model of random motion used to model the motion of molecules and fluctuations of the stock market, among other phenomena. We focus on Brownian motion in two dimensions.

Brownian path starting at the origin



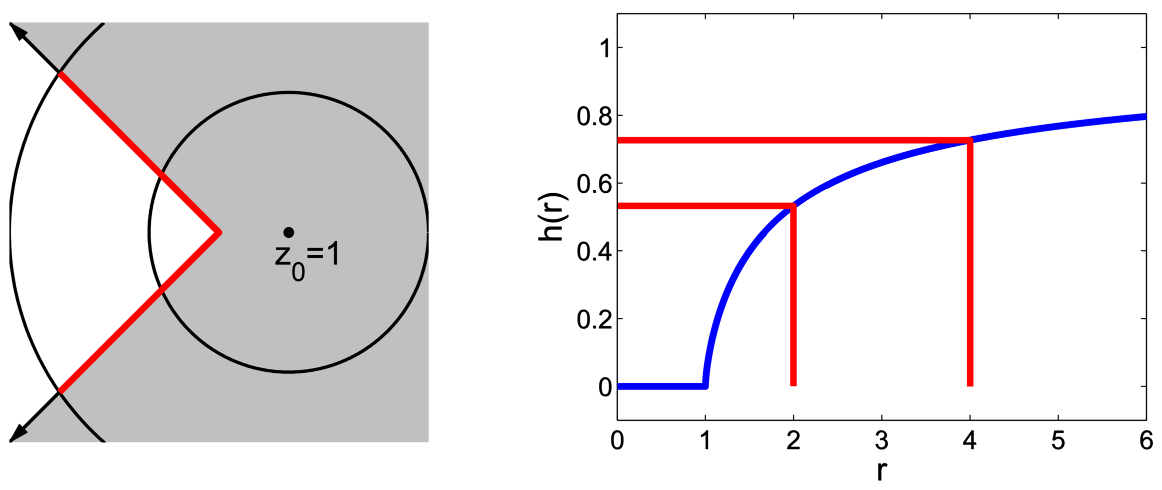
h-functions

The harmonic measure distribution function, or h-function, of a domain gives information about how a Brownian particle moves in that domain. Given a domain such as the shaded region below, we start a Brownian particle moving from the basepoint z_0 until it hits the boundary of the domain. For each fixed radius $r > 0$, the h-function $h(r)$ is defined to be the probability that the particle hits the boundary within distance r of the basepoint, that is, on the red portion of the boundary in the figure below.



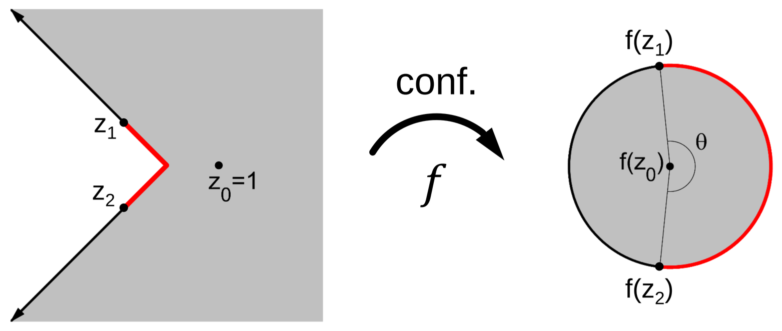
Properties of h-functions

- The h-function is always between 0 and 1 because it represents a probability.
- The h-function $h(r)$ is 0 for all $r < r_0$, where r_0 is the shortest distance from the basepoint z_0 to the boundary of the domain.
- The h-function is increasing with r. If $R > r$, there is a greater probability that the particle will it the boundary of the domain within distance R of z_0 than within distance r of z_0, so $h(R) > h(r)$.
- The h-function tends towards 1 as r increases because we are guaranteed that the Brownian particle will eventually hit the boundary of the domain.



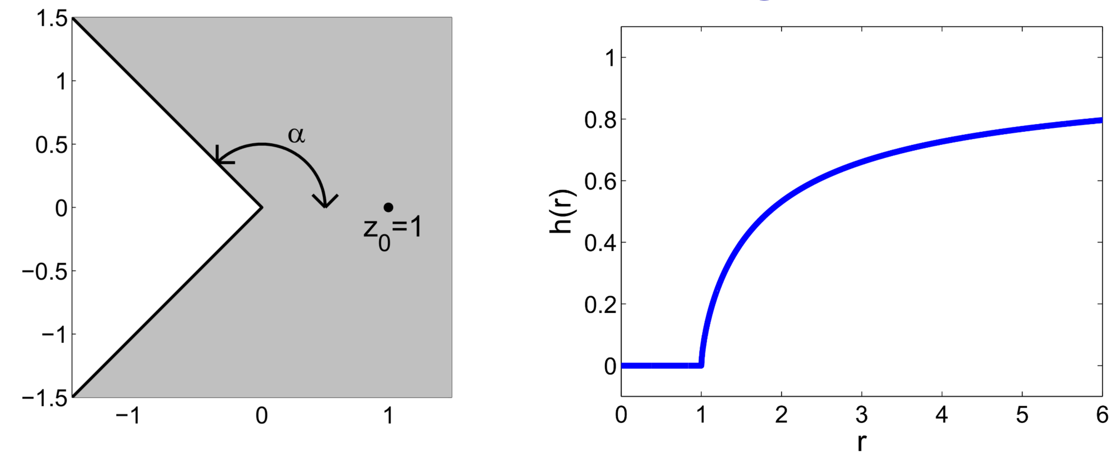
Computing h-functions Exactly

If there exists a conformal (angle-preserving) map from a domain to the interior of the unit disk, then we can compute the h-function exactly. In the example below, f is a conformal map from the domain on the left to the unit disk.

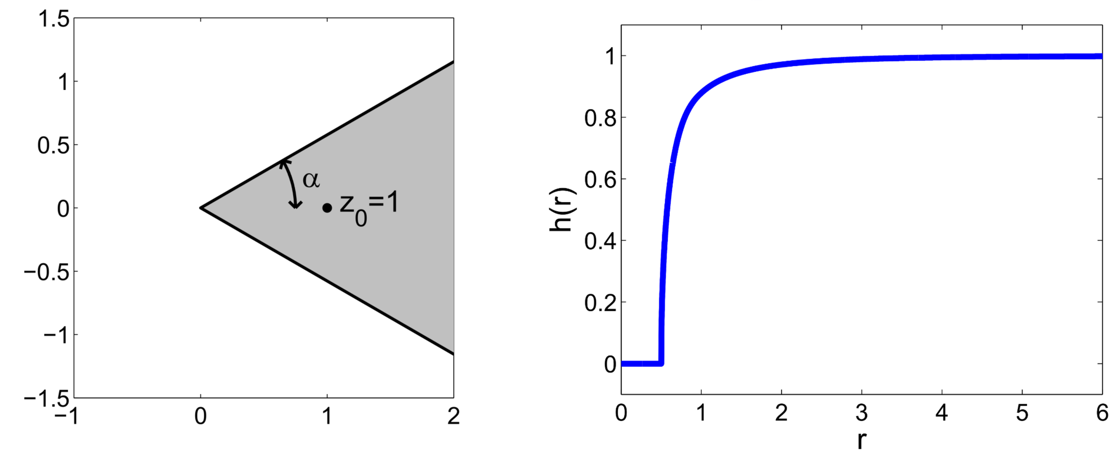


The probability that a Brownian particle starting at z_0 will first hit the red portion of the boundary in the domain on the left is equal to the probability that a Brownian particle starting at $f(z_0)$ will first hit the red portion of the boundary in the domain on the right. This probability is given by $\frac{\theta}{2\pi}$.

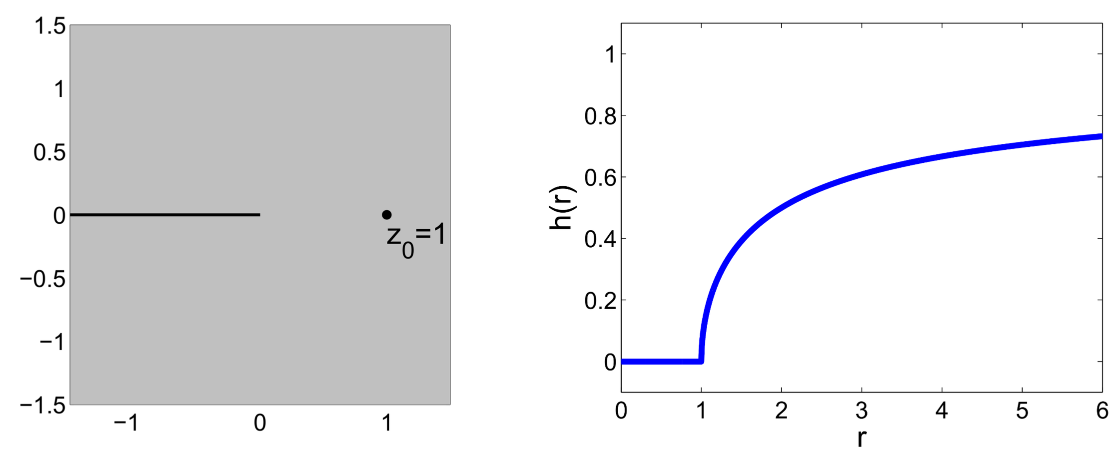
Exterior of Wedge



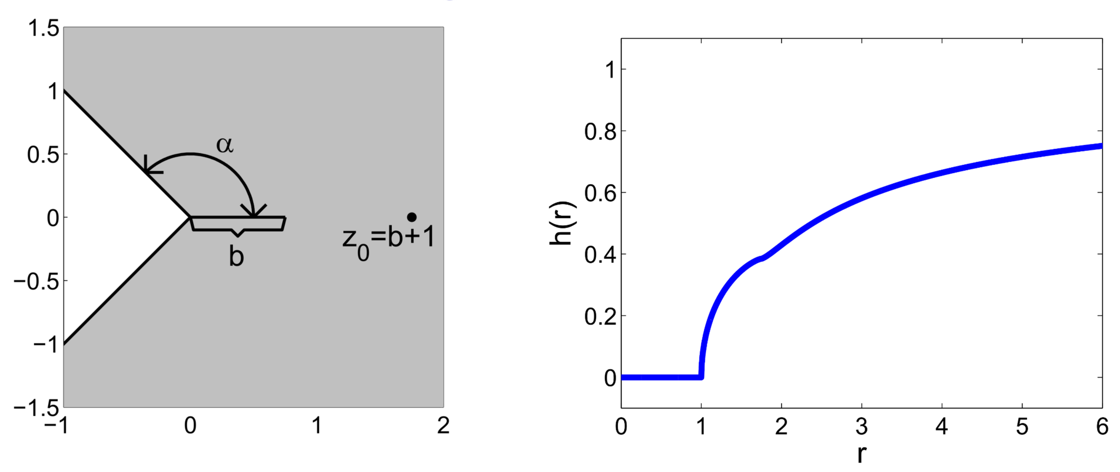
Interior of Wedge



Slit Plane



Wedge with Spike



Derivatives of h-functions

- For the interior of wedge domain, the derivative $h'(r)$ is continuous at $r = 1$.
  - This means the rate at which a Brownian particle “sees” the boundary does not change abruptly when the radius reaches the corner.
  - The second derivative $h''(r)$ is continuous at $r = 1$ when $0 < \alpha < \frac{\pi}{4}$, but discontinuous when $\frac{\pi}{4} < \alpha \leq \frac{\pi}{2}$.
- For the wedge with spike domain, the derivative $h'(r)$ is continuous at $r = b + 1$, and $h'(b + 1) = 0$.

Convergence of h-functions

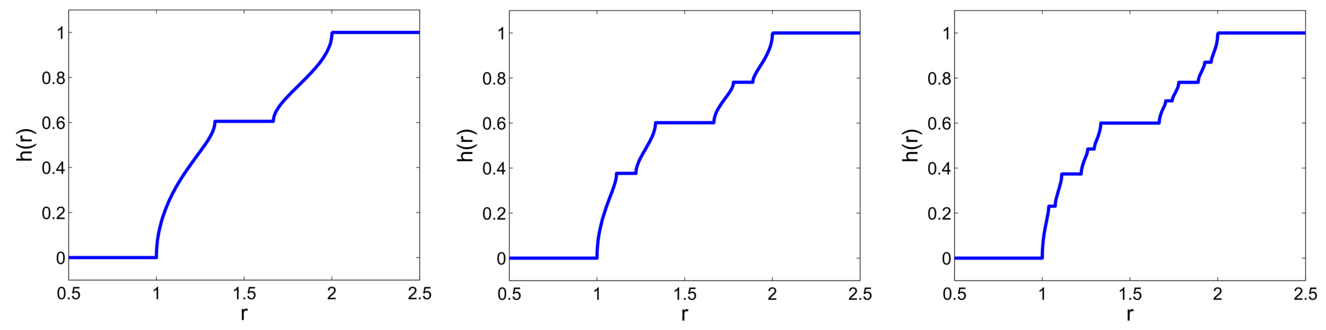
- I constructed a sequence of exterior of wedge domains with their angles increasing toward π. I also constructed a sequence of wedge with spike domains with their spike lengths increasing toward infinity.
- The domains in both sequences look more and more like the slit plane.
- I proved that the sequences of h-functions for both of these sequences of domains converge pointwise to the h-function for the slit plane. That is, the h-functions get closer to the h-function for the slit plane.
- For the exterior of wedge domains, I extended this result to uniform convergence on any interval $[r^*, \infty)$ with $r^* > 0$.

Simulations

The *Cantor set* is built in stages. Starting with the interval $[0,1]$, the middle third of the interval, $\left(\frac{1}{3}, \frac{2}{3}\right)$, is removed, leaving the intervals $\left[0, \frac{1}{3}\right]$ and $\left[\frac{2}{3}, 1\right]$. Next the middle thirds of these two intervals are removed, and so on.

Constructing the Cantor Set

The *Cantor domain* is the complex plane without the Cantor set, with basepoint $z_0 = -1$, so its boundary consists of just the Cantor set. It can be approximated in stages by using stages of the Cantor set. I ran simulations of Brownian motion in Matlab to approximate the h-functions for the first several stages of the Cantor domain.



Conjectures

- For the interior of wedge domain with angle $\alpha < \frac{\pi}{2}$, the n^{th} derivative $h^{(n)}(r)$ is continuous at $r = 1$ if $0 < \alpha < \frac{\pi}{2n}$ but discontinuous if $\alpha \geq \frac{\pi}{2n}$.
- For the wedge with spike domain, $h''(r)$ is discontinuous at $r = b + 1$.
- These conjectures imply that corners in a domain can be detected in discontinuities in some derivative of the h-function.
- Whenever a sequence of h-functions converges pointwise to an h-function, the convergence is uniform.
- The height of the middle step of the h-function for each stage of the Cantor domain decreases monotonically as the stage increases. This could be useful in finding the exact value of the step height.

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