# Searching for and Characterizing Abundancy Outlaws

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## Abstract

For a positive integer $n$, the abundancy index $I(n)$ is defined to be the sum of its divisors divided by the number itself, or $\sigma(n)/n$. The function $I : \mathbb{N} \to \mathbb{Q} \cap (1, \infty)$ is not onto; rationals not in the range of $I$ are called "abundancy outlaws." Identifying and characterizing abundancy outlaws could prove helpful to better understand the existence of odd perfect numbers, a question over 2000 years old. In our research, we consider rationals of the form $(\sigma(n)+t)/n$, where $t$ is a positive integer, to produce and characterize as-yet undiscovered outlaws.

## Introduction

The study of the abundancy index is motivated by interest in perfect numbers. A positive integer is a *perfect number* if it is equal to the sum of its proper divisors. The smallest perfect number is $6 = 1 + 2 + 3$, followed by 28, 496, and 8128. Euclid showed that even numbers of the form $2^{p-1}(2^p - 1)$ are perfect when $p$ and $2^p - 1$ are prime, while Euler later showed that every even perfect number must have this form. Thus we have a complete characterization for even perfect numbers: finding one is equivalent to finding prime $p$ such that $2^p - 1$ is prime. Currently there are 48 known perfect numbers—all of which are even. As a search to $10^{300}$ found no odd perfect numbers, the question is whether it can be proved that no odd perfect numbers exist.

The abundancy index has been studied as a means to find such a proof, which is defined by $I(n) = \sigma(n)/n$, i.e., the ratio of the sum of the divisors of $n$ $\sigma(n) = \Sigma_{d|n} d$ and $n$ itself. An integer $n$ having $I(n) = 2$ is perfect; for example, $I(6) = (1 + 2 + 3 + 6)/6 = 2$; integers having other integer-valued indices are multiperfect. "Deficient" $n$ have index less than 2 and "abundant" $n$ have non-integer index greater than 2. Thus, the abundancy index measures the "perfection" of a number.

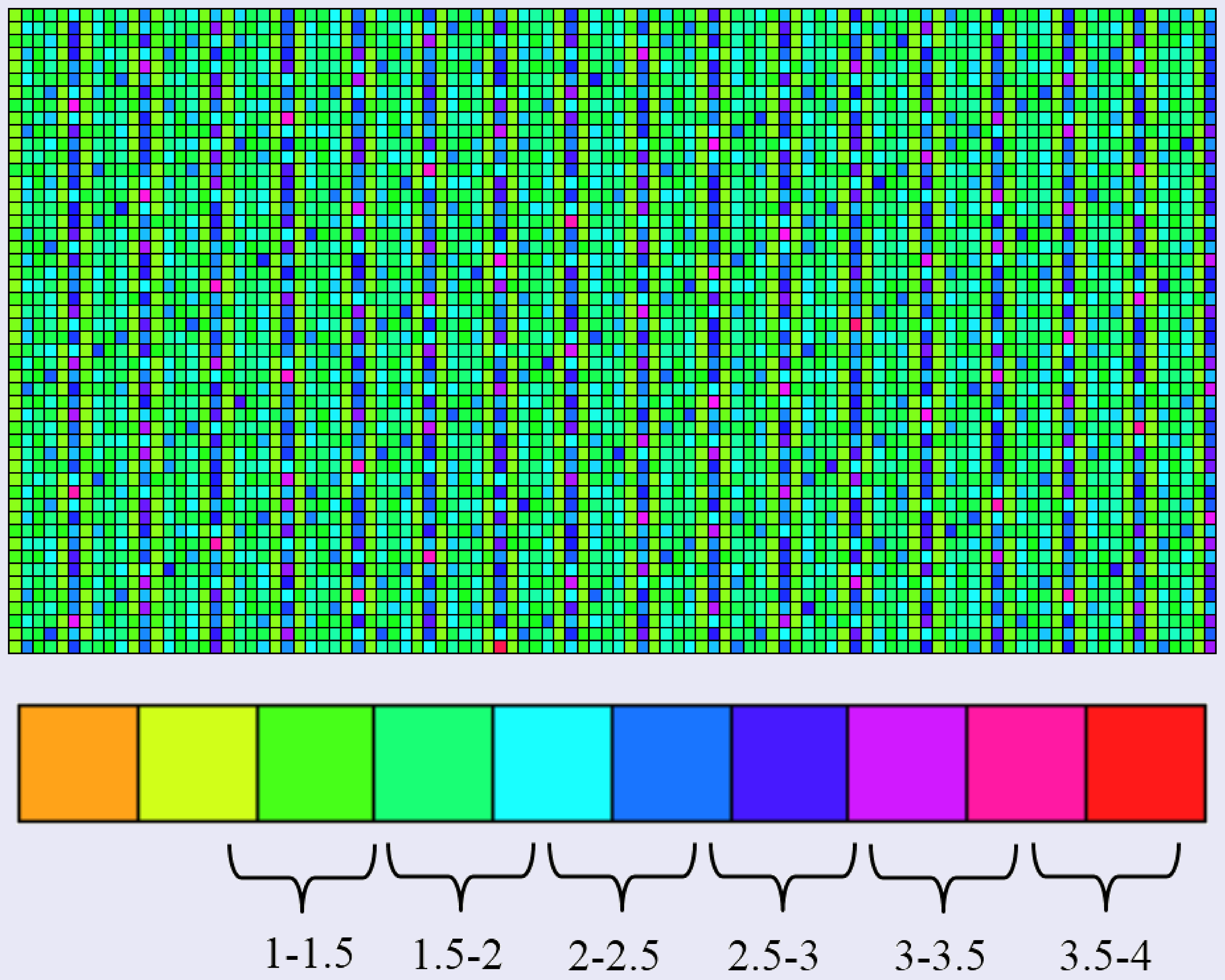


Figure 1: Abundancy index of the first 5,100 integers coded by color

## Abundancy Outlaws

As $I : \mathbb{N} \to \mathbb{Q} \cap [1, \infty)$ is not onto, rationals not in the range of $I$ are termed *abundancy outlaws*. Finding outlaws is particularly important as the existence of an odd perfect number is equivalent to the existence of an abundancy index of a particular form [1]:

**Theorem:** There exists an odd perfect number if and only if there exist positive integers $p, n$, and $\alpha$ such that $p \equiv \alpha \equiv 1 \pmod 4$, where $p$ is a prime not dividing $n$, and

$$I(n) = \frac{2p^\alpha(p-1)}{p^{\alpha+1} - 1}.$$

For example, if there exists $n$ such that $I(n) = 5/3$, then $5n$ is an odd perfect number. Thus, a proof that such rationals are abundancy outlaws would prove that no odd perfect number exists.

Rationals of the form $(\sigma(N) + t)/N$ with $t$ negative were characterized by Paul Erdős, who proved that if $(k, m) = 1$ and $m < k < \sigma(m)$, then $k/m$ is an abundancy outlaw. For example, $5/4$ is an outlaw because $(5, 4) = 1$ and $4 < 5 < \sigma(4) = 7$.

Our research searched for outlaws of the same form with positive $t$.

## Searching for Outlaws

The main search method utilized the theorem below, which requires the following definition:

**Definition:** If $m$ and $n$ are positive integers, let $m_n$ denote the largest divisor of $m$ each of whose prime factors divide $n$.

**Example:** $60_{10} = (2^2 \cdot 3 \cdot 5)_{2 \cdot 5} = 2^2 \cdot 5 = 20$

**Theorem:** Given a positive integer $N$, let $v$ be such that $I(v)$ is minimal given the following:

(i) $N | v$

(ii) $\sigma(v)_N | (v/N)$.

If $t$ is a positive integer smaller than $I(v)N - \sigma(N)$ with $(\sigma(N)+t, N) = 1$, then $(\sigma(N) + t)/N$ is an abundancy outlaw.

| $N$ | $\sigma(N)$ | $v/N$ | $I(v)$ | $t <$ | Outlaws |
|---|---|---|---|---|---|
| $12 = 2^2 \cdot 3$ | 28 | 3 | 91/36 | 2.3 | 29/12 |
| $20 = 2^2 \cdot 5$ | 42 | 5 | 217/100 | 1.4 | 43/20 |
| $24 = 2^3 \cdot 3$ | 60 | $2^3$ | 127/48 | 3.5 | 61/24 |
| $30 = 2 \cdot 3 \cdot 5$ | 72 | $3 \cdot 5$ | 403/150 | 8.6 | 73/30, 77/30, 79/30 |
| $40 = 2^3 \cdot 5$ | 90 | 5 | 93/40 | 3 | 91/40 |
| $42 = 2 \cdot 3 \cdot 7$ | 96 | $3 \cdot 7^3$ | 36413/14406 | 10.2 | 97/42, 101/42, 103/42 |
| $45 = 3^2 \cdot 5$ | 78 | 5 | 403/225 | 2.6 | 79/45 |
| $48 = 2^4 \cdot 3$ | 124 | $2^2$ | 127/48 | 3 | 125/48 |
| $56 = 2^3 \cdot 7$ | 120 | 7 | 855/392 | 2.1 | 121/56 |
| $60 = 2^2 \cdot 3 \cdot 5$ | 168 | $3 \cdot 5$ | 2821/900 | 20.1 | 169/60, 173/60, 179/60, 181/60, 187/60 |
| $66 = 2 \cdot 3 \cdot 11$ | 144 | $3 \cdot 11$ | 1729/726 | 13.2 | 145/66, 149/66, 151/66, 155/66, 157/66 |
| $70 = 2 \cdot 5 \cdot 7$ | 144 | $5 \cdot 7$ | 5301/2450 | 7.5 | 149/70, 151/70 |
| $72 = 2^3 \cdot 3^2$ | 195 | 2 | 403/144 | 6.5 | 197/72, 199/72 |
| $78 = 2 \cdot 3 \cdot 13$ | 168 | $3 \cdot 13^3$ | 30941/13182 | 15.1 | 173/78, 175/78, 179/78, 181/78 |
| $80 = 2^4 \cdot 5$ | 186 | 2 | 189/80 | 3 | 187/80 |
| $84 = 2^2 \cdot 3 \cdot 7$ | 224 | $3 \cdot 7$ | 247/84 | 23 | 227/84, 229/84, 233/84, 235/84, 239/84, 241/84 |
| $88 = 2^3 \cdot 11$ | 180 | 11 | 1995/968 | 1.4 | 181/88 |
| $90 = 2 \cdot 3^2 \cdot 5$ | 234 | $3^2 \cdot 5$ | 3751/1350 | 16.1 | 239/90, 241/90, 247/90 |
| $96 = 2^5 \cdot 3$ | 252 | $2^3$ | 511/192 | 3.5 | 253/96 |
| $99 = 3^2 \cdot 11$ | 156 | 11 | 1729/1089 | 1.2 | 157/99 |

Figure 2: Outlaws of the form $(\sigma(N) + t)/N$ for $N < 100$, with newly discovered outlaws in purple.

## Characterizing Outlaws

In the results of our main search technique, we observe several patterns. In particular, we are able to characterize an upper bound on $t$ given certain conditions on $N$:

**Theorem:** Let $p$ be a prime and $\alpha \geq 1$ be odd. For integers $m$ and $t$, assume there exists prime $q | p+1$ not dividing $mt$, and assume $(\sigma(p^\alpha m) + t, p^\alpha m) = 1$. If $t < \sigma(m)/p$, then $\frac{\sigma(p^\alpha m)+t}{p^\alpha m}$ is an outlaw.

**Theorem:** Let $p$ be a prime and $\alpha \geq 2$ be even. For integers $m$ and $t$, assume $(\sigma(p^\alpha), mt) = 1$ and that $(\sigma(p^\alpha m)+t, p^\alpha m) = 1$. If there exists prime $q < p$ dividing $\sigma(p)$ but not $mt$, then $\frac{\sigma(p^\alpha m)+t}{p^\alpha m}$ is an outlaw for $t < (1/p + 1/p^2)\sigma(m)$. Otherwise $\frac{\sigma(p^\alpha m)+t}{p^\alpha m}$ is an outlaw for $t < \sigma(m)/p$.

**Theorem:** Assume $N = \Pi_{i=1}^{k} p_i^{\alpha_i}$, with $k > 1$, and suppose $t$ is a positive integer satisfying:

(i) $(\sigma(N) + t, N) = 1$ and $(t, \sigma(N)) = 1$

(ii) $t < \min\left\{\frac{1}{p_j}\Pi_{i \neq j}\sigma(p_i^{\alpha_i})\right\}_{j=1}^{k}$

(iii) $t < \sigma(\sigma(N)) - \sigma(N)$.

Then $\frac{\sigma(N)+t}{N}$ is an outlaw.

The three results each find a number of additional outlaws.

## Limitations and Further Research

The main search technique is limited when $(N, \sigma(N)) = 1$, $\sigma(N)_N = 1 | (n_N/N)$. Then $v = N$ and $t < I(N)N - \sigma(N) = 0$, so we find no outlaws. In particular, no outlaws are found for prime $N$ as $(\sigma(N) = N+1, N) = 1$ Thus we are able to say nothing about rationals of the form $\frac{\sigma(p)+1}{p} = \frac{p+2}{p}$, which have long eluded characterization. Thus it seems new methods of searching for outlaws will be required to, for example, determine whether $5/3$ is an outlaw.

Our characterizations of outlaws also runs into difficulty when $N$ is divisible by 2 and 3, again suggesting the need for new machinery with which to discover abundancy outlaws.

### *Acknowledgements and References*

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[1] Holdener, Judy. "Conditions Equivalent to the Existence of Odd Perfect Numbers." *Mathematics Magazine*, Vol. 79, No. 5. (Dec., 2006), pp. 389-291.

[2] Stanton, Will and Judy Holdener. "Abundancy 'Outlaws' of the Form $\frac{\sigma(N)+t}{N}$," *Journal of Integer Sequences*, Vol 10 (2007), Article 07.9.6.

[3] Weiner, Paul. "The Abundancy Ratio, a Measure of Perfection". *Mathematics Magazine*, Vol. 73, No. 4. (Oct., 2000), pp. 307-310.

[4] Laatsch, Richard. "Measuring the Abundancy of Integers". *Mathematics Magazine*, Vol. 59 (1986), pp. 84-92.