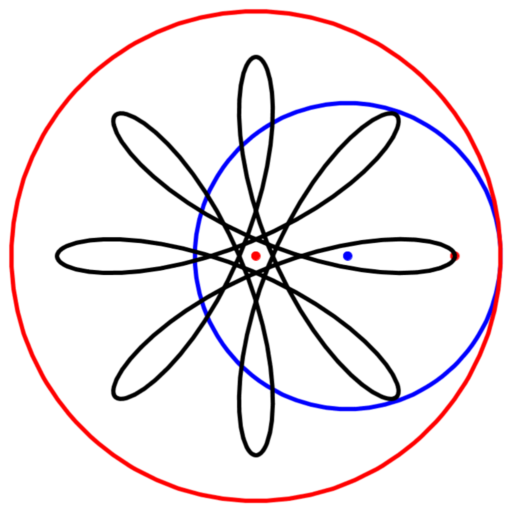
TANGLOIDS AND THEIR SYMMETRIES

Seth Colbert-Pollack & Prof. Carol S. Schumacher
Department of Mathematics, Kenyon College, Gambier, OH 43022

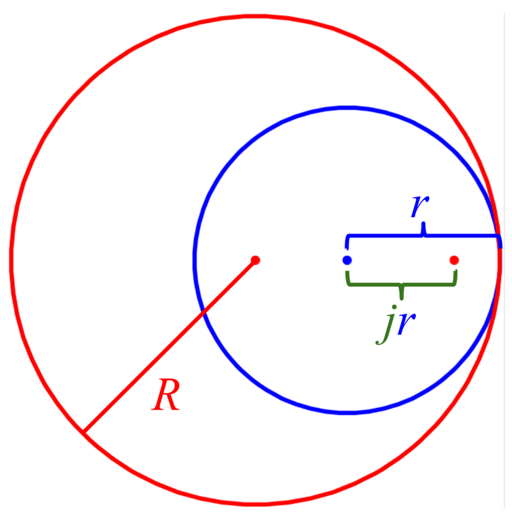
HYPOTROCHOIDS

Hypotrochoids are the types of curves drawn by a Spirograph. They are made by tracing a point on a circle (the "pen") as it rolls around the inside of another circle.



We trace the right red point as the blue circle rolls around the red one.

Hypotrochoids are described using three parameters:



- R, the radius of the fixed circle.
- r, the radius of the rolling circle.
- j, which tells us how far the pen is from the center of the rolling circle.

The hypotrochoid at top has $R = 8$, $r = 5$, and $j = 0.7$.

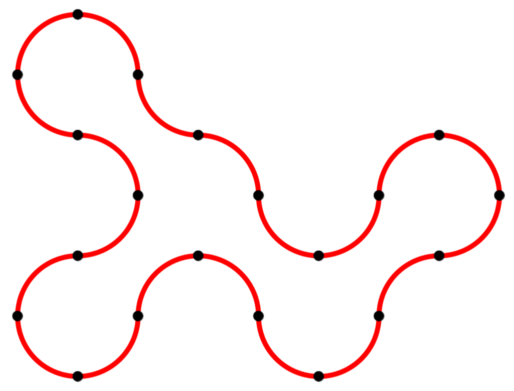
HYPOTROCHOID PROPERTIES

Hypotrochoids have properties which we later generalize for tangloids. For example, we know that the curve closes up if and only if $\frac{R}{r}$ is rational. More specifically, if $\frac{R^*}{r^*}$ is $\frac{R}{r}$ expressed in lowest terms, then

- R^* is the number of "petals" of the curve; i.e. its degree of rotational symmetry.
- r^* is the number of times the rolling circle must roll around the fixed circle before the curve closes up. (We call this quantity the "minimum number of revolutions to closure.")

TANGLES

A **tangle** is a closed curve composed of quarter circles of equal radii. (There must be a multiple of 4 quarter circles if the curve is to "close up.")

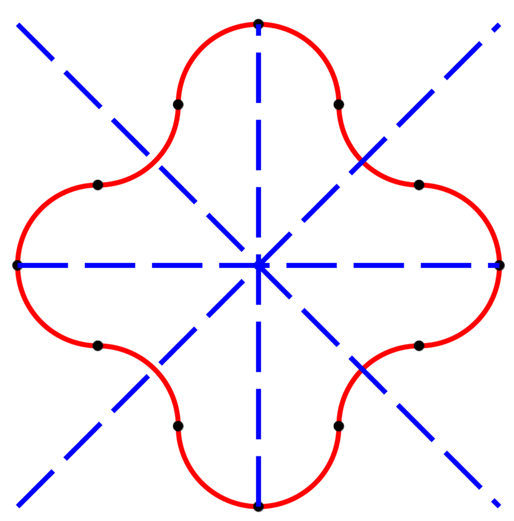


A tangle of length 20.

We are interested in the curves created by rolling a circle around these shapes.

TANGLE PROPERTIES

- We can describe each tangle by a list P of 1s and -1s, representing left and right turns respectively. P will always sum to 4 or -4.
- A P is not unique to a tangle; a tangle can be represented starting at different quarter circles, leading to different Ps.
- **Rotational Symmetry**: A tangle can have only π or $\frac{\pi}{2}$ rotational symmetry.
  - Tangles with π rotational symmetry are described by a P that is composed of two identical blocks.
  - For $\frac{\pi}{2}$ rotational symmetry, we need four identical blocks.
- **Reflectional Symmetry**: A tangle has reflectional symmetry if and only if it can be represented by a P which is a palindrome.



This tangle has $P = [1, -1, 1, 1, -1, 1, 1, -1, 1, 1, -1, 1]$. It has $\frac{\pi}{2}$ rotational symmetry as well as 4 lines of reflectional symmetry.

TANGLOIDS

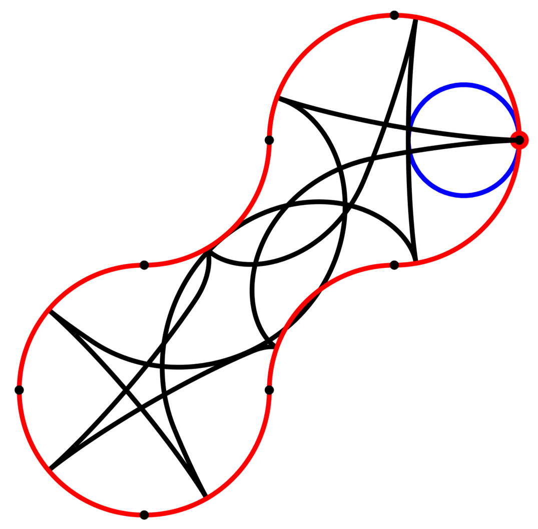
We define a **tangloid** as the curve generated by following a point on a circle as the circle rolls around a tangle.

- We use the same parameters as with hypotrochoids to describe the tangloid, except R now refers to the radii of the quarter circles that make up the tangle.
- $\frac{R^*}{r^*}$ is still $\frac{R}{r}$ in lowest terms.

As for hypotrochoids, the tangloid will be a closed curve if and only if $\frac{R}{r}$ is rational. Furthermore, we proved the following theorem:

Theorem. For a tangle of length $4z$, the minimum number of revolutions to closure for the resultant tangloid is

$$\frac{\operatorname{lcm}(z, r^*)}{r^*}.$$



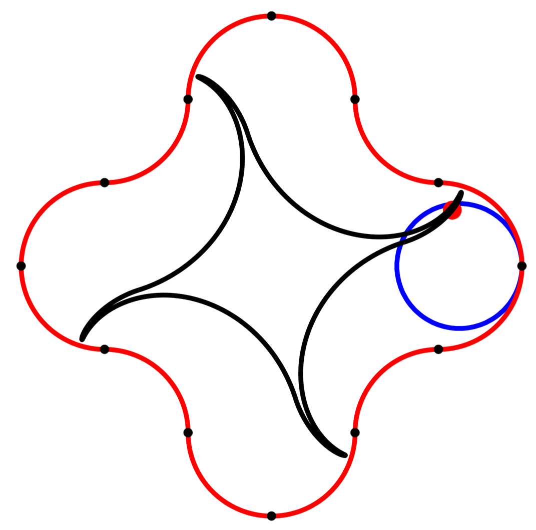
This tangloid has $R = 9, r = 4, j = 1$, and $z = 2$, so it has 2 revolutions to closure.

ROTATIONAL SYMMETRIES

- We seek to generalize our hypotrochoid rule for r^* to tangloids.
- Unsurprisingly, tangloids which have rotational symmetry can only be formed from tangles with the same symmetry.
- Using our known properties of P for a tangle that has a certain rotational symmetry, we proved the following theorem:

Theorem. Given a tangle of length $4z$ with π or $\frac{\pi}{2}$ rotational symmetry, its tangloid has respectively:

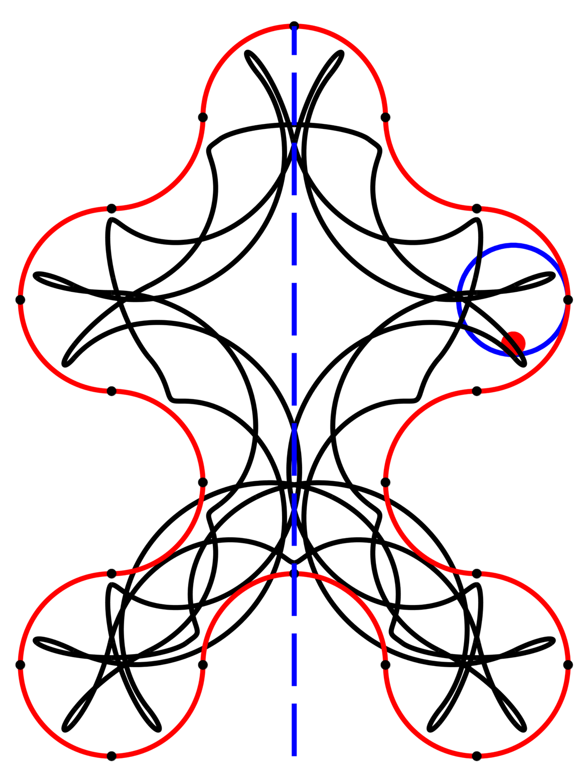
- π rotational symmetry if and only if $\frac{zR^*}{r^*}$ is a multiple of 2.
- $\frac{\pi}{2}$ rotational symmetry if and only if $\frac{zR^*}{r^*}$ is a multiple of 4.



A rotationally symmetric tangloid of $R = 4, r = 3, j = 0.9$, and $z = 3$.

REFLECTIONAL SYMMETRIES

- Like rotational symmetries, tangloids with reflectional symmetry can only be formed from tangles with the same symmetry.
- Reflectional symmetries are highly dependent on the initial position of the pen inside the rolling circle.
- This initial position is given by the positive angle made by the radius through the pen with the x-axis.
- Indexing the tangloids resulting from all the possible initial pen positions gives us a "family of tangloids."
- We conjecture that if the tangle of a tangloid has only one line of symmetry, then exactly two members of that tangloid's family have that symmetry.



We set the initial position of the pen on the rolling circle such that the tip of a petal is on a line of symmetry.

REFERENCES

[1] Fleron, Julian F. *The Geometry of Model Railroad Train Tracks and the Topology of Tangles: Glimpses into the Mathematics of Knot Theory via Children's Toys*. Private communication, 2000.
[2] Schumacher, Carol. "Analysis of the Hypotrochoid." Vector analysis lab, 2009.

FUTURE RESEARCH

We would like to prove our conjecture about reflectional symmetries. Beyond that, we are interested in the roulettes of other tangle-like objects. For example, a tangle is made up of links of quarter circles, but what about shapes made up of links of third circles? Given one of these shapes of length $3z$ has $\frac{2\pi}{3}$ rotational symmetry, can we conclude that its roulette will have the same symmetry if and only if $\frac{zR^*}{r^*}$ is a multiple of 3?

ACKNOWLEDGEMENTS

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