Characterization of the Positive Integers with Abundancy Index of the Form (2x-1)/x

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Abstract

The abundancy index of a positive integer n is defined to be the rational number $\sigma(n)/n$ where $\sigma(n)=\sum_{\{d|n\}} d$ is the sum of the divisors of n, and $I: N \to \mathbb{Q} \cup (1,\infty)$ is the function defined by $I(n)=\sigma(n)/n$. Erdös showed that I is not onto, and if $m/n>1$ fails to fall in the range of I, then mn is called an “abundancy outlaw.” In this research we examined the form of positive integers N satisfying $I(N)=(2x-1)/x$, where x is a positive integer. Rational numbers of the form $(2x-1)/x$ are important since both even and odd perfect numbers have a divisor with abundancy index of this form. We discovered that there are many even integers satisfying the condition, and we characterized some patterns exhibited by such even integers. Among the odd integers, however, we were only able to identify one odd integer N less than 10^8 satisfying $I(N)=(2x-1)/x$.

Introduction

A perfect number is a natural number equal to the sum of all its proper divisors, e.g. $6, 28, 496, 8128$…The symbol $\sigma(n)$ is used to represent the sum of all the divisors of a natural number n including itself.

For perfect numbers: $\sigma(n)=2n$

$\sigma(6)=1+2+3+6=2(6)$

$\sigma(28)=1+2+4+7+14+28=2(28)$

Perfect numbers have intrigued many mathematicians from ancient time. The first record of it appears on *Euclid’s Elements*. It terms out the characterization of this type of number is not very straightforward. In fact, there are two major open problems in mathematics relating to perfect numbers.

Even Perfect Numbers

- Are there infinitely many even perfect numbers?
- Euclid proved over 2000 years ago that if 2^p-1 is a Mersenne prime, then $2^{p-1}(2^p-1)$ is an even perfect number.
- Later Euler proved that if N is an even perfect number, then it has the form $2^{p-1}(2^p-1)$, completely characterizing even perfect numbers.
- There are 49 known Mersenne primes and hence 49 known even perfect numbers. The largest known even perfect number is: $2^{274,207,280}(2^{274,207,281}-1)$.
- This question is equivalent to whether there are infinitely many Mersenne primes.

Odd Perfect Numbers

- Are there any odd perfect numbers?
- In 1953, Touchard proved that an odd perfect number must be of the form $2k+1$ or $36k+9$.[5]
- It has been checked computationally for odd numbers up to 10^{300} without success.
- Nielson (2006) proved that an odd perfect number must have at least 9 distinct prime factors.
- Euler gave a famous characterization of odd perfect numbers

Theorem If N is an odd perfect number, then N must have the form

$$N=p^{\alpha}{p_1}^{2\alpha_1}{p_2}^{2\alpha_2}\cdots{p_r}^{2\alpha_r}$$

where $p, p_1, p_2, \ldots, p_r$ are primes and $p=\alpha=1(mod\ 4)$.

Abundancy of Integers

One useful tool of studying perfect numbers is abundancy index. The *abundancy index* of a positive integer n is the rational number $I(n)=\sigma(n)/n=\sum_{d|n} d/n$. By definition, the abundancy index of a perfect number is 2. In fact, the abundancy index is a function I mapping the set of natural numbers to the rationals $I: \mathbb{N} \to \mathbb{Q} \cap [1,\infty)$. In the 1970’s, Erdös proved that the map is not surjective. For example, $5/4=(2^2+1)/2^2$ is not in the image of I. Hence, we define an *abundancy outlaw* is a rational number not in the image of the map I. It has been proved that both the sets of abundancy indexes and abundancy outlaws are dense in the interval $(1,\infty)$, which adds complexity to the problems related to perfect numbers.

Theorem (Holdener 2007)[1] There exists an odd perfect number iff there exist positive integers p, n and α such that $p=\alpha=1(mod\ 4)$, where p is a prime not dividing n, and

$$I(n)=\frac{2p^{\alpha}(p-1)}{p^{\alpha+1}-1}$$

This theorem indicates that understanding the sets of abundancy indices and outlaws is important for exploring questions relating to perfect numbers.

Motivation

- We investigated rationals of the form $\frac{\sigma(n)+t}{n}$ where $t>0$, because rationals of the form k/n where $n<k<\sigma(n)$ are already known to be outlaws.
  - For example, consider $5/3=(\sigma(3)+1)/3$
- Hard Open Problem: Is $5/3$ an index or an outlaw?[4]

  If $\frac{\sigma(n)}{n}=\frac{5}{3}$ for some n, then $\frac{\sigma(5n)}{5n}=\frac{\sigma(5)\sigma(n)}{5n}=\frac{6}{5}\cdot\frac{5}{3}=2.$

  So $5n$ is an odd perfect number.

Observe that $\frac{5}{3}=\frac{2(3)-1}{3}=\frac{2x-1}{x}$ where $x=3$

- Ryan (2003) also explored the rationals in this form and here is his results [3].

Theorem Suppose we have a fraction of the form $(2x-1)/x$, where $2x-1$ is prime. *i)* If x is even, but not a power of 2, then $(2x-1)/x$ is an outlaw. *ii)* If n is odd and $I(b)=\frac{2x-1}{x}$, then b is odd; moreover, if $2x-1$ does not divide b then $b(2x-1)$ is a perfect number.

Abundancy of the Form $I(N)=\frac{2x-1}{x}$

Here are some characterization of the integers N such that $I(N)=(2x-1)/x$. We first consider the case where N is even.

- **Case 1 (One prime factor):**

When $N=2^m$ for some positive integer m,

$I(N)=I(2^m)=\frac{2^{m+1}-1}{2^m}=\frac{2\cdot 2^m-1}{2^m}=\frac{2\cdot 2^m-1}{x}$ where $x=2^m$.

All $N=2^m$ satisfy the equation! (2^m is almost perfect.)

- **Case 2 (Two distinct prime factors):**

We prove that if $N=2^m p^k$ for some positive integers m and k and $I(N)=(2x-1)/x$ for some $x\in\mathbb{N}$, then $k=1$.

- **Case 3 (More that two distinct prime factors):**

  This case gets complicated!

We ended up examining integers of the form $N=2^m p_1 p_2\cdots p_n$ where $2<p_1<p_2<\cdots<p_n$.

Abundancy of the Form $I(N)=\frac{2x-1}{x}$

m	Some $n=2^m{p_1}^{k_1}{p_1}^{k_2}\cdots{p_n}^{k_n}<10^8$ with $1\le m\le 5$ satisfying $I(n)=\frac{2x-1}{x}$
1	$(2)(5)$
2	$(2)^2(11), (2)^2(13)(17), (2)^2(11)^2(29)(197)$
3	$(2)^3(17), (2)^3(19), (2)^3(23),$ $(2)^3(19)(113), (2)^3(17)(137), (2)^3(17)(139), (2)^3(17)^2(307)$ $(2)^3(37)(41)(73), (2)^3(37)(47)^2(61), (2)^3(27)(149)(1489)$
4	$(2)^4(47)$ $(2)^4(59)(67), (2)^4(41)(131), (2)^4(41)(163), (2)^4(37)(197)$ $(2)^4(37)(199), (2)^4(41)(163)(653)$
5	$(2)^5(67), (2)^5(71), (2)^5(79),$ $(2)^5(109)(151), (2)^5(97)(193), (2)^5(79)(317),$ $(2)^5(71)(569), (2)^5(71)(571), (2)^5(71)(709), (2)^5(67)(1607)$

Theorem: Suppose $n\ge 2$ and $p_1<p_2<\cdots<p_n$ are odd primes. If $N=2^m p_1 p_2\cdots p_k$ satisfies $I(N)=\frac{2x-1}{x}$ for some $x\in\mathbb{N}$, then for all $1\le k\le n$.

$$k\sigma(2^m)<p_k$$
$$<\sigma(2^m)\sigma(p_1)\cdots\sigma(p_{k-1})(n-(k-1))+2^m p_1 p_2\cdots p_{k-1}$$

Now let’s look at the case where N is odd…

We have only found one odd example by searching for N up to 10^8: $N=3^2 7^2 11^2 13^2$

- $3^2 7^2 11^2 13^2$ is a square.
- In fact, it’s easy to prove that if N is odd and $I(N)=(2x-1)/x$, then N is a square.
- Take a close look at $I(3^2 7^2 11^2 13^2)$…

$$I(3^2 7^2 11^2 13^2)=\frac{\sigma(3^2)}{3^2}\frac{\sigma(7^2)}{7^2}\frac{\sigma(11^2)}{11^2}\frac{\sigma(13^2)}{13^2}$$
$$=\frac{13}{3^2}\cdot\frac{3\cdot 19}{7^2}\cdot\frac{7\cdot 19}{11^2}\cdot\frac{3\cdot 61}{13^2}$$
$$=\frac{19^2\cdot 61}{7\cdot 11^2\cdot 13}=\frac{2(7\cdot 11^2\cdot 13)-1}{7\cdot 11^2\cdot 13}=\frac{22021}{11011}$$

Furfure Exploration

- Explore the pattern for the more general case

  $$N=2^m{p_1}^{k_1}{p_2}^{k_2}\cdots{p_n}^{k_n}$$

- The odd case:
  - Are there any more odd N such that $I(N)=\frac{2x-1}{x}$? Can we characterize them?
  - Why are odd cases so rare?

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