Tangloids: A Mathematical Exploration into Children's Toys

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Abstract

Spirograph curves are small plastic toys that allow one to make colorful shapes and intricate patterns, grabbed the attention of many children and even some mathematicians. These forms (also known as Hypotrochoids and Epitrochoids) have inherent mathematical qualities that have been studied in depth. However, it is unknown what the properties of these forms are if the outer circle isn't a circle at all, but rather the forms were fixed curves of differing shapes and forms. Our research set out to investigate these properties by using Tangles, another children's toy. These objects are quarter circles which can create closed curves of many different types, containing sections of both Hypotrochoids and Epitrochoids. Using these tangles as the larger curves, we used the Spirograph technique to study curves which are created with a circle and a pen going around the curves. Appropriately, we named these Tangloids. Our results enable one to generalize the curves of Hypotrochoids and Epitrochoids and their properties. We defined the conditions under which Tangloids had rotational symmetry and the conditions under which they had reflection symmetries, generalizing similar properties of Hypotrochoids.

Background

Epitrochoids are made by moving a smaller circle around the outside of a larger circle and Hypotrochoids are created by moving a smaller circle on the inside of a larger circle. Below you'll see the types of curves they often make and the "petals" they usually have. We know that Hypotrochoids and Epitrochoids are closed curves, meaning their curves close up and don't continue infinitely, if and only if the radius of the larger circle over the radius of the smaller circle is a rational number. Furthermore, the reduced form of big R, defined as the radius of the larger circle over little r, defined as the radius of the smaller circle, has the property that the denominator is equal to the number of times the small circle must go around the big circle. These properties are well known and have been studied in depth.

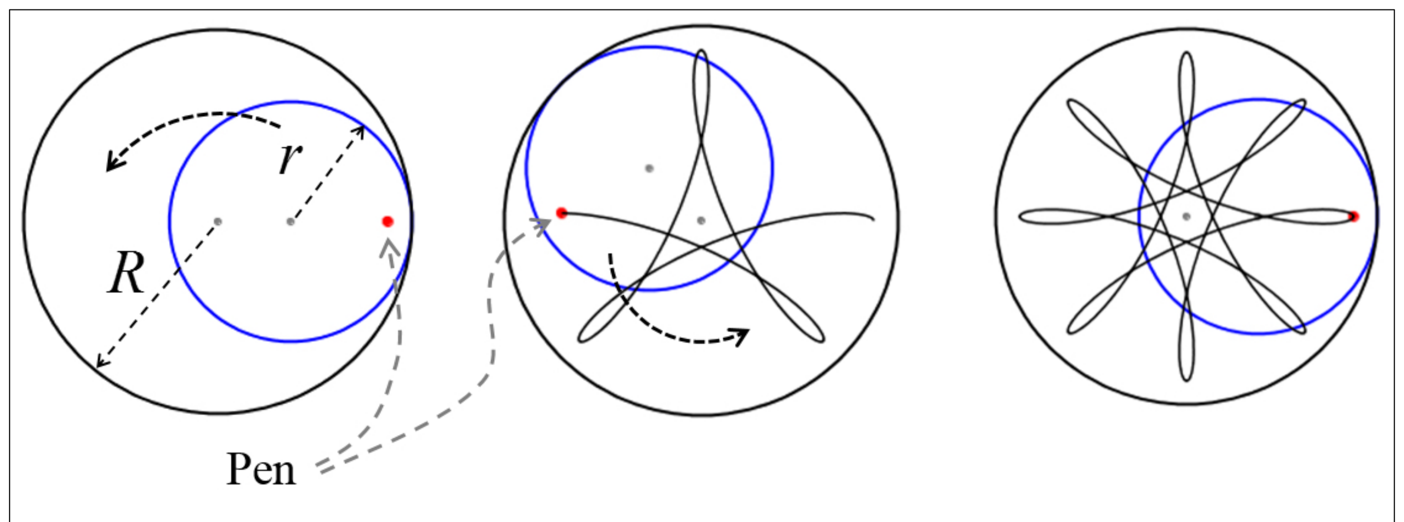
Equation for an Epitrochoid:

$$x(t) = (R+r)\cos(t) + jr\,(\cos(t - \tfrac{R}{r}t)$$
$$y(t) = (R+r)\sin(t) + jr\,(\sin(t - \tfrac{R}{r}t)$$

Equation for a Hypotrochoid:

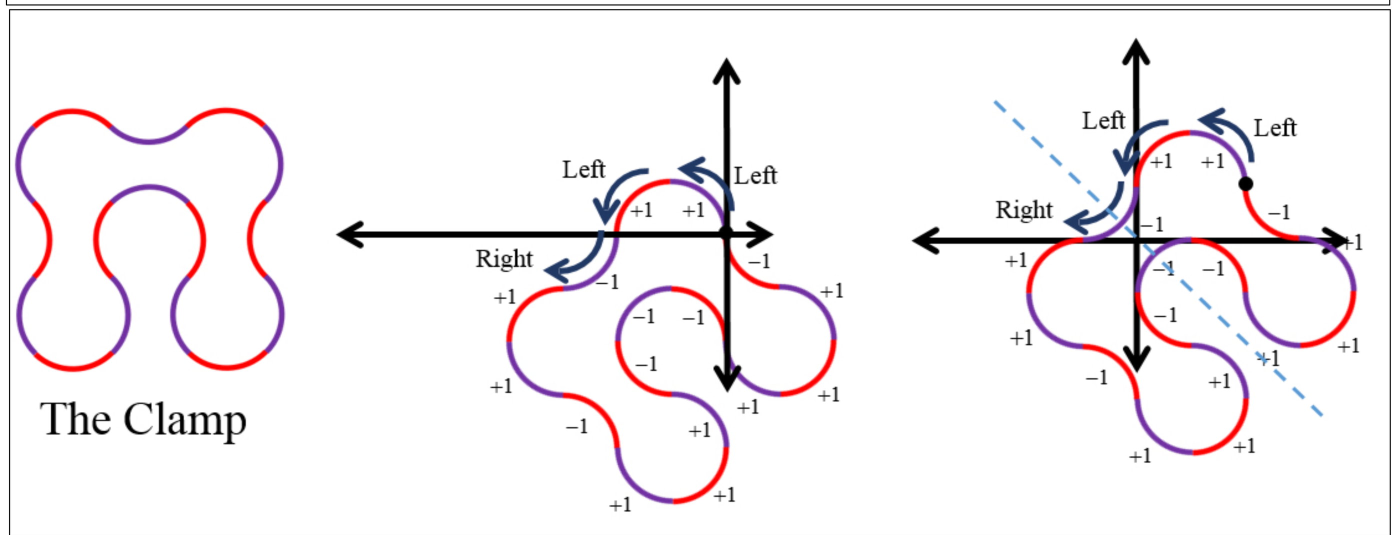
$$x(t) = (R-r)\cos(t) + jr\,(\cos(t - \tfrac{R}{r}t)$$
$$y(t) = (R-r)\sin(t) + jr\,(\sin(t - \tfrac{R}{r}t)$$

Where:
- The variable R is the radius of the larger circle.
- The variable r is the radius of the smaller circle
- The variable j is the percentage distance from the center of the circle to the edge, or the position of the pen.
- Both equations are functions of t, time.



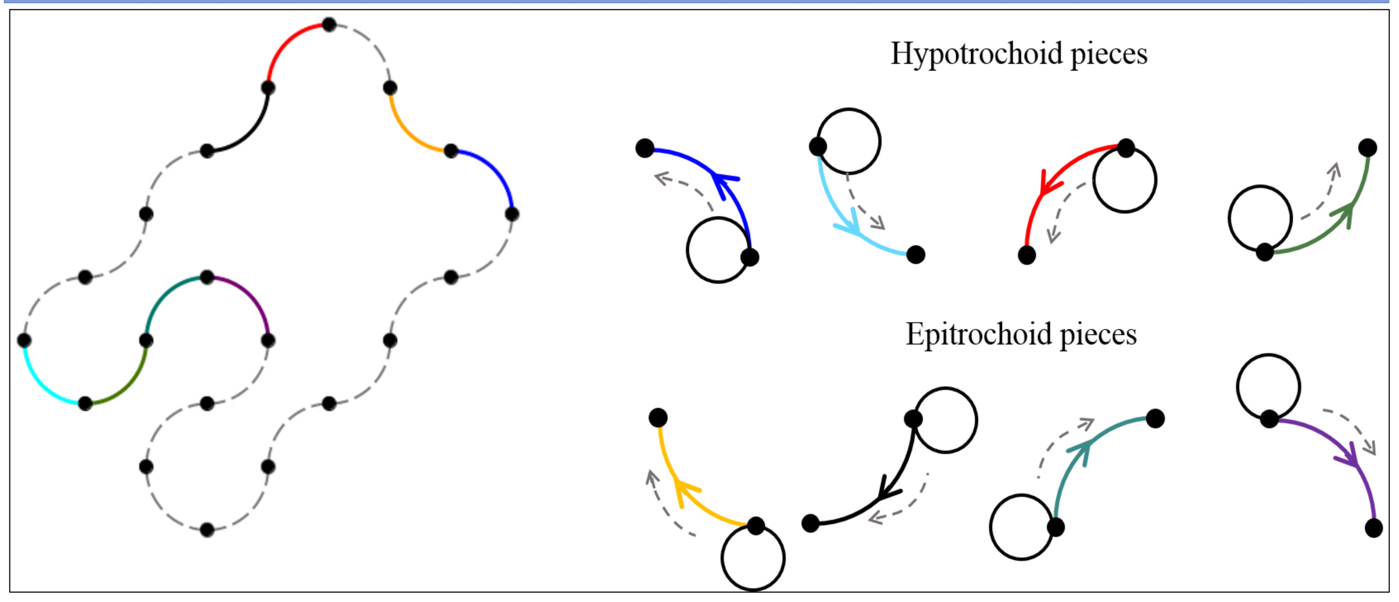
Describing Tangles

Tangles are quarter circles that snap together to make different shapes. A Tangle can be made to lie flat only if the number of quarter circles comprising it is divisible by 4. In order to parametrize Tangloids we had to come up with a method for describing tangles.



Each tangle can be described using a sequence of +1s and -1s.

Tangloid Properties



Once we set the wheel turning inside our tangle, we see that a Tangloid is made up of Hypotrochoid pieces (+1s), and Epitrochoid pieces (-1).

Tangloid Equation

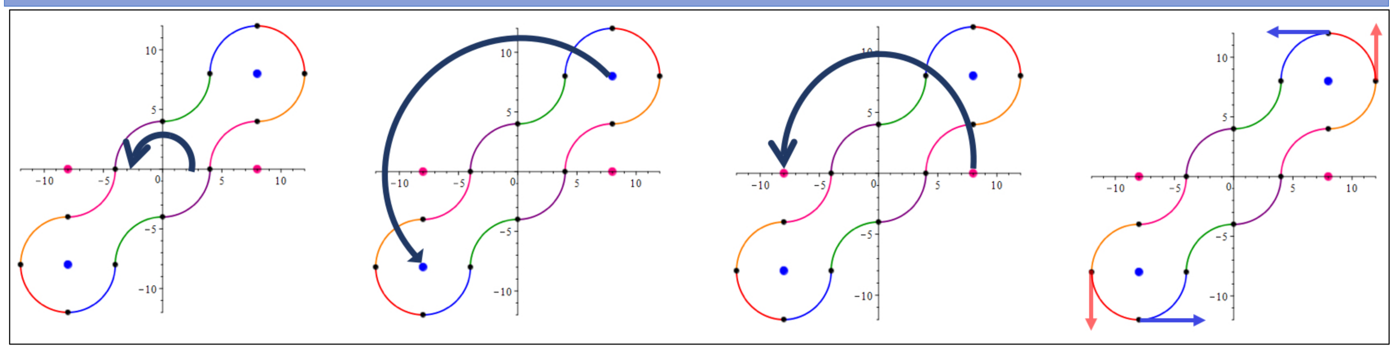
A circle is made up of four quarter circles, and thus is a tangle. Thus hypotrochoids are a special case of Tangloids What we realize from this is that any discoveries we make about Tangloids become generalizations of the specific properties of hypotrochoids. Equation that we used for the kth Tangloid piece:

$$TC_k(\theta) = C_k + (R - t_k r)\begin{bmatrix}\cos(t_k\theta + a_k)\\ \sin(t_k\theta + \alpha_k)\end{bmatrix} + \text{pr}\begin{bmatrix}\cos(t_k\theta - \frac{R}{r}\theta + \tau_k)\\ \sin\left(t_k\theta - \frac{R}{r}\theta + \tau_k\right)\end{bmatrix}$$

Where:
- k is each tangle piece
- C is the center of the larger circle that a tangle piece is a part of
- t is the tangle value (+1 or -1)
- theta is the time
- tau is the location of the pen at the beginning of the kth tangle piece
- alpha tells us where the smaller circle is in relation to the larger circle
- p is the position of the pen in the smaller circle

Each of these variables has their own specific equations.

Rotational Symmetries



Big Question: If a tangle has rotational symmetry, do the corresponding Tangloids share that symmetry?

Expanding on the research done last summer by Seth Colbert Pollack and Professor Schumacher, we were able to show that *some do* and *some don't*. Here we focus on π-rotational symmetry. (180º degree rotational symmetry.)

We observe that if a Tangloid *TC* has π-rotational symmetry then for some integer v:

$$F_\pi(TC_k) = TC_{k+nv+\frac{n}{2}}$$

Where F_π is the rotational matrix and n is the number of pieces in the tangle.

This knowledge would lead us to our proof. Here is one essential part of the proof:

$$\frac{\pi}{2}\left(nv + \frac{n}{2}\right)\frac{R}{r} = 2\pi l$$
$$\frac{1}{4}\left(nv + \frac{n}{2}\right)\frac{R}{r} = l$$
$$\frac{nR}{4r}\left(j + \frac{1}{2}\right) = l$$
$$\frac{nR}{4r} = \frac{l}{\left(j + \frac{1}{2}\right)}$$
$$\frac{nR}{4r} = \frac{2l}{(2j+1)}$$

In summary, we concluded that for this pi rotational symmetry to exist in a Tangloid, $\frac{nR}{4r}$ where
- n is equal to the number of tangle pieces
- big R is the radius of the tangle pieces
- little r is equal to the radius of the little circle

had to be reduced to $2l$ over $2j+1$ where l and j are arbitrary integers. So essentially, Tangloids have pi rotation symmetry when $\frac{nR}{4r}$ is equivalent to an even value over an odd value.

Reflection Symmetries

Big Question: If a tangle has reflection symmetry, do the corresponding Tangloids share that symmetry?

Using the equation for the timer (which is the placement of the pen at a certain k) we concluded that a tangle would have rotational symmetry if and only if the timer started out at the correct value.

For Reflection Symmetries to hold:

$$TC_{f+j+1} = Rev(TC_{f+nv-j})$$

Where:
- f and j are arbitrary integers
- nv is the number of revolutions needed to close up the curve.

The correct value for τ_1 means that the pen must be at the same angle as the line of symmetry when it hits that line. This can have many different variations.

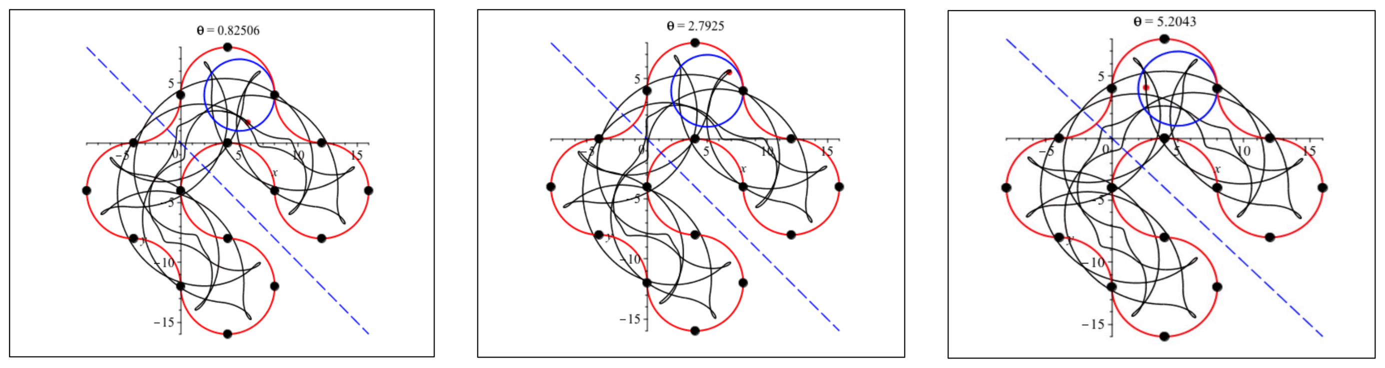
In order to prove this we had to use the equation for the timer:

$$\tau_{k+1} = \tau_k + t_k\frac{\pi}{2} - \frac{R}{r}\frac{\pi}{2} \text{ and thus } \tau_k = \tau_1 + \frac{\pi}{2}\sum_{i=1}^{k-1} t_i - \frac{R}{r}\frac{\pi}{2}(k-1)$$

Tangloid Family

Looking at the reflection symmetries Tangloids, one observation we made is that there are certain "families" of curves. These curves all are the same, with some changes when a certain part gets placed down. We proved that when the initial angle τ_1 was moved up by $\frac{2\pi s}{nv}$ it yields the same curve.

Below you'll see an example of a Tangloid that can have various curves based on where τ_1 is.



Further Questions

- What happens when we place the smaller circle on the outside of the tangle?
- What happens when tangle pieces have different radii?
- What does it mean when the Tangloid crosses over itself and how is that related to the "petals" in hypotrochoids?

References

- Ippolito, Dennis. "The Mathematics of the Spirograph." *The Mathematics Teacher*, vol. 92, no. 4, 1999, pp. 354–358. *JSTOR*, JSTOR, www.jstor.org/stable/27970991.

Acknowledgments

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